香港青少年數學精英選拔賽

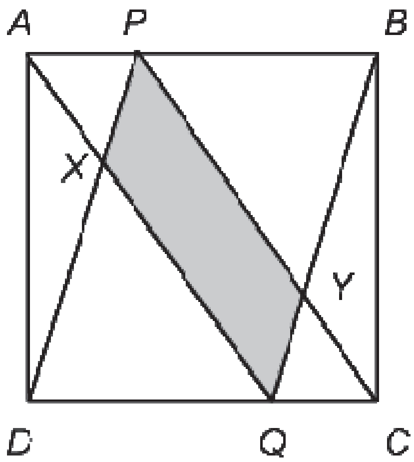
The Hong Kong Mathematical High Achievers Selection Contest

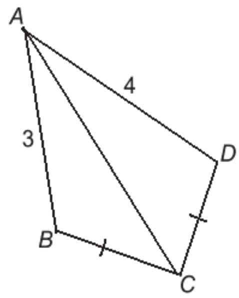
2014 – 2015

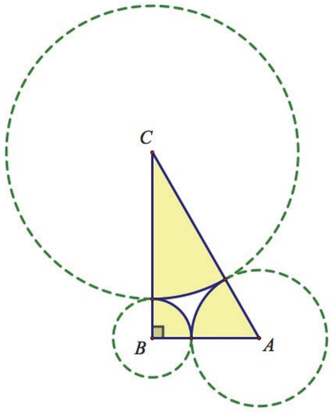
甲部 **(每題 2 分)**

把答案填在答題紙所提供的位置。

1	若 a 及 b 為兩個不大於 2015 的正整數。求 $\dfrac{a+b}{a-b}$ 的最大值。 Find the greatest value of $\dfrac{a+b}{a-b}$, where a and b are positive integers not greater than 2015.
2	若 a、b、c 為整數，且 $abc = 2015$，求 $a+b+c$ 的最小值。 If a, b and c are integers and $abc = 2015$, find the smallest value of $a+b+c$.
3	如圖，小明由學校步行回家。及後，老師發現小明留下鎖匙在學校，就立即去追小明，將鎖匙交給他後立即折返回校，再過 30 分鐘，小明回到家中，而老師亦同時回到學校。若老師的速率是小明的三倍，那麼，小明由學校步行回家需時多少？  As shown in the figure, Dennis walks from school to home. When his teacher finds that Dennis has left his key at school, his teacher rushes to give him back the key and then back to school immediately. After 30 minutes, Dennis arrives home and his teacher arrives school. If the speed of his teacher is three times that of Dennis, then how long it will take Dennis to walk from school to home?
4	若 y 為整數，$x=\sqrt{124+\sqrt{2015}}$ 且 $x \le y$，求 y 的最小值？ If y is an integer, $x=\sqrt{124+\sqrt{2015}}$ and $x \le y$, find the least value of y.

5	已知 31^x=2015 及 65^y=2015，求 $\frac{1}{x}+\frac{1}{y}$ 的值。 If 31^x=2015 and 65^y=2015, find the value of $\frac{1}{x}+\frac{1}{y}$.
6	將 *x* 本書分給 100 名學生，每名學生至少分得一本，且沒有 4 名學生分得同數目的書，求 *x* 的最小值。 Divide *x* books among 100 students such that each student receives at least one book and no four students obtain same number of books. Find the least value of *x*.
7	將 990 個邊長為 1 cm 的小立方體結合成為一個 9cm ×10cm ×11cm 長方體。隨後把長方體的表面著色。問原來的 990 個小立方體中有多少個是只有一面著了色？ We glue together 990 small cubes with side 1 cm into a 9cm ×10cm ×11cm rectangular solid. Then we paint the outside of the solid. Find the number of the original 990 cubes having just one of their sides painted.
8	*ABCD* 是一個面積為 1 平方單位的正方形。設 *P* 及 *Q* 分別在 *AB* 及 *CD* 上，使得 $AP = CQ < 0.5$。設 *X* 為 *AQ* 及 *DP* 的交點，*Y* 為 *BQ* 及 CP 的交點，若四邊形 *PXQY* 的面積為 0.21，求 *AP*。  In a square *ABCD* of area 1 square unit, let *P* and *Q* be points on *AB* and *CD* such that $AP = CQ < 0.5$. Let *X* be the point of intersection of *AQ* and *DP*, and *Y* be the point of intersection of *BQ* and *CP*. If the area of quadrilateral *PXQY* is 0.21 , find *AP*.

9	求一邊長為 29 而兩對角線長相差 2 的菱形的面積。 Find the area of a rhombus whose side length is 29 and whose diagonals differ in length by 2.
10	設 *ABCD* 為凸四邊形，$AB = 3$, $AC = 5$, $AD = 4$, $BC = CD$ 及 $\angle BCD = 60°$ 。 求該四邊形的面積。  Let *ABCD* be a convex quadrilateral with $AB= 3$, $AC= 5$, $AD= 4$, $BC = CD$ and $\angle BCD = 60°$. Find the area of the quadrilateral.
11	已知 2015 個連續正整數的和是一個平方數，其中 x 為最大的數。求 x 的最小值。 It is given that the sum of 2015 consecutive integers is a square number, x is the greatest number among the integers, find the value of x.
12	求 $\sqrt{1+2015\sqrt{1+2014\sqrt{1+2013\sqrt{1+2012\sqrt{1+2011\times 2009}}}}}$ 的值。 Find the value of $\sqrt{1+2015\sqrt{1+2014\sqrt{1+2013\sqrt{1+2012\sqrt{1+2011\times 2009}}}}}$.
13	已知 $\frac{a}{a^2+a+1}=\frac{1}{4}$ ，求 $\frac{a^2}{a^4+a^2+1}$ 的值。 It is given that $\frac{a}{a^2+a+1}=\frac{1}{4}$, find the value of $\frac{a^2}{a^4+a^2+1}$.
14	求所有為 96 或 180 的正因數的和。 Find the sum of all positive factors of either 96 or 180.
15	將數字填入以下方格中使得每三個連續方格的數字和為 2015。求最左邊方格填入的數字。 Fill in numbers in the boxes below so that the sum of the entries in each three consecutive boxes is 2015. Find the number that goes into the leftmost box. <table><tr><td></td><td></td><td>999</td><td></td><td></td><td></td><td></td><td>888</td><td></td></tr></table>

16	在圖中，ΔABC 是一個角為 30°、60° 及 90° 的三角形，當中 $AC = 20$。三個分別以三角形的三個頂點為圓心的圓，每個圓外切另外兩個圓。三個圓與三角形重疊的三個扇形的面積和為 $(m+n\sqrt{3})\pi$，其中 m 與 n 為有理數。求 $m+n$。  In the figure, ΔABC is a triangle with angles 30°, 60°, 90° and $AC = 20$. Three circles, each externally touching the other two, have centres at the three vertices of the triangles. The sum of the areas of three sectors inside the triangle is $(m+n\sqrt{3})\pi$ for some rational numbers m and n. Find $m + n$.
17	$ABCD$ 為一凸四邊形，面積為 1。延長 AB 至 B' 使得 $BB' = AB$；延長 BC 至 C' 使得 $CC' = BC$；延長 CD 至 D' 使得 $DD' = CD$；延長 DA 至 A' 使得 $AA' = DA$。求四邊形 $A'B'C'D'$ 的面積。 $ABCD$ is a convex quadrilateral with area 1 square unit. AB is produced to B' with $BB' = AB$; BC is produced to C' with $CC' = BC$; CD is produced to D' with $DD' = CD$; DA is produced to A' with $AA' = DA$. Find the area of the quadrilateral $A'B'C'D'$.
18	求在實數 x-y 平面上的所有能滿足下列方程系的點： $\begin{cases} x^2 + y^2 = 1 \\ (x+\sqrt{1+x^2})(y+\sqrt{1+y^2}) = 1 \end{cases}$ Find **all** points on the real x-y plane that satisfying the following system of equations : $\begin{cases} x^2 + y^2 = 1 \\ (x+\sqrt{1+x^2})(y+\sqrt{1+y^2}) = 1 \end{cases}$

乙部 **(每題 6 分)**

把完整的題解和答案寫在答題紙所提供的位置。

19 A、B 及 C 三隊進行乒乓球賽。

規則如下：

首先由 A、B 兩隊比賽。每場由兩隊各派出一人比賽，勝方與其餘一隊派出的一人比賽，並依次進行比賽 (例如若每隊 3 人，編號為 A1、A2、A3；B1、B2、B3；C1、C2、C3。A1 勝 B1 的話，A1 就對 C1，若 A1 又勝 C1 的話，A1 就對 B2 等)。若其中一隊的隊員全部出局的話，則剩下的兩隊繼續比賽直至最後一場的勝方為冠軍隊。

(a) 若每隊 2 人，則冠軍隊最少勝多少場比賽？

(b) 若每隊 2015 人，則冠軍隊最少勝多少場比賽？

Three teams A, B and C are in the process of participating in table tennis competition.

The Rules are as follows:

First one member from team A and team B compete, then the winner will compete with the remaining team in the order of each team (For example, when there are three members in each team, namely A_1, A_2, A_3 ;B_1, B_2, B_3; C_1, C_2, C_3. If A_1 wins B_1 , then A_1 vs C_1, if A_1 wins again , then A_1 vs B_2 and so on). In case, all members of one team lose, the remaining two teams will continue to compete till the last remaining team to the Championship.

(a) If there are two members in each team, find the minimum number of rounds a team must wins in the competition in order to compete for the championship.

(b) If there are 2015 members in each team, find the minimum number of rounds a team must wins in the competition in order to compete for the championship.

20 一個有 N 個委員的提名委員會現提名競選總統的候選人，每個委員最多可提名 k 人 (無需是提名委員會的委員)。 只有當某人得到超過半數的委員提名，他或她才能成功成為候選人。 填滿下表，表中每個數值代表對於一對已知的 N 及 k 的最大可能候選人的數目， 而 $2 \le N \le 15$， $1 \le k \le 6$ 。

A nominating committee of N persons is to nominate candidates for the post of President, each person nominating at most k names (not necessarily that of a member of the nominating committee). A name becomes a nominated candidate only if he or she earns endorsement from more than half of the committee. Fill in the following table where the entry is the largest number of possible nominated candidates for a given pair of N, k, with $2 \le N \le 15$, $1 \le k \le 6$.

k \ N	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1														
2														
3														
4														
5														
6														

21 假設在實數 x - y 平面上能滿足方程

$$(x+1+x^2)(y+1+y^2)=1$$

的點形成曲線 C。

(a) 在 C 上的所有點中，求具有最大 y 坐標的一點。試解釋你的答案。

(b) 在 C 上的所有點中，求具有最小 x 坐標的一點。試解釋你的答案。

(c) 求 r 的一個值，使得曲線 C 完全落在於半徑為 r 及圓心為原點的圓之內。試清楚解釋你如何選擇 r 的值。

Suppose all points on the real *x-y* plane that satisfying the equation

$$(x+1+x^2)(y+1+y^2)=1$$

form a curve C.

(a) Among all the points on C, find a point which has the largest y coordinate. Explain your answer.

(b) Among all the points on C, find a point which has the smallest x coordinate. Explain your answer.

(c) Find one value of r so that the curve C is lying entirely inside a circle with radius r and centre at the origin. Justify your choice of r carefully.

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