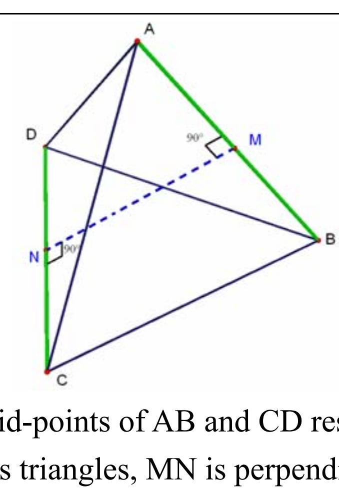
香港青少年數學精英選拔賽

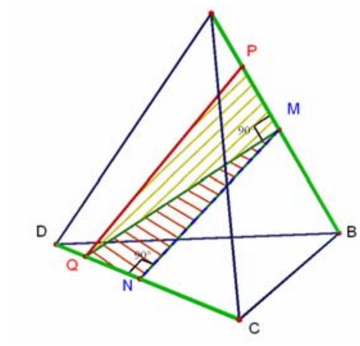
The Hong Kong Mathematical High Achievers Selection Contest

2007 – 2008

SUGGESTED SOLUTION

No.	Answer	Explanation
1.	120	$\because 143 = 11 \times 13$ ∴the number of reduced proper fractions = 143 – 13 – 11 + 1 = 120
2.	$\frac{1}{2009}$	$\frac{(2008^2 - 2008 - 6)(2008^2 + 2\times 2008 - 3)}{2005\times 2007\times 2009\times 2010\times 2011}$ $= \frac{(2008-3)(2008+2)(2008+3)(2008-1)}{2005\times 2007\times 2009\times 2010\times 2011}$ $= \frac{1}{2009}$
3.	c > d > b > a	$\frac{b}{a} = \frac{15^{25}}{3^{50}} = \frac{15^{25}}{9^{25}} > 1$; $\frac{d}{b} = \frac{1024^{10}}{15^{25}} = \frac{16^{25}}{15^{25}} > 1$; $\frac{c}{d} = \frac{33^{20}}{1024^{10}} = \frac{33^{20}}{32^{20}} > 1$;
4	18 and 16	Let x and y be the length and breadth respectively. $2(x + y) = xy$ $2x = xy - 2y$ $y = \frac{2x}{x-2} = 2 + \frac{4}{x-2}$. Therefore 4 must be divisible by x – 2. Thus x – 2 can be 1, 2 or 4. Hence, the corresponding values of y are 6, 4 or 3. From the above results, the possible areas of the rectangles are 18 and 16.
5	80°	Let $\angle BAE = x$ Then $\angle FAD = x$ and $\angle ABE = \frac{180^o - x}{2}$ Since $AD // BC$, $\angle ABE + \angle BAE + \angle EAF + \angle FAD = 180^o$ $\frac{180^o - x}{2} + x + 60^o + x = 180^o$, $\therefore x = 20^o$ Therefore $\angle ABE = 80^o$
6	AB can be 16cm, 20cm, 14cm or 22cm	Let $AB = BC = x$ and $AC = y$ Case(i) $AB + BD = 24$ and $AC + CD = 30$ $x + \frac{x}{2} = 24$ and $y + \frac{x}{2} = 30$ ∴ x = 16, AB = 16cm Case(ii) $AB + BD = 30$ and $AC + CD = 24$ $x + \frac{x}{2} = 30$ and $y + \frac{x}{2} = 24$ ∴ x = 20, AB = 20cm Also, consider AC = BC. Similarly, we have AB = 14cm or 22cm.

7	$-\frac{1}{6}$	$a_1=-\frac{1}{6}$, $a_2=7$, $a_3=\frac{6}{7}$, $a_4=-\frac{1}{6}$,… The sequence will be repeated every 3 terms. When 2008 is divided by 3, the remainder is 1. $\therefore a_{2008}=a_1=-\frac{1}{6}$.
8	27.5	Extend AD and BC to meet at the point E Since $\angle A=90^o$ and $\angle B=45^o$, so $\angle E=45^o$ Therefore, ΔABE and ΔCDE are isosceles. So $AE=8$ and $CE=3$ $\therefore$ the area of the convex polygon = $\frac{1}{2}\times 8\times 8-\frac{1}{2}\times 3\times 3=27.5$
9	23	Since Mr Chan is an adult, so he must be born on $\overline{19xy}$. This year is 2008, so $2008-\overline{19xy}=1+9+x+y$ $2008-1900-10x-y=1+9+x+y$ $98=11x+2y$ $x=8+\frac{10-2y}{11}$, $\therefore y=5$ and $x=8$ Mr Chan was born on 1985 and he is 23 years old now.
10	446	$\frac{2}{7}=0.\dot{2}8571\dot{4}$ The sum of 2, 8, 5, 7, 1 and 4 is 27 When 2008 is divided by 27, the quotient is 74 and the remainder is 10. $\therefore$ the value of n is 74 x 6 + 2 = 446.
11	26	Put a = $\sqrt{7}+\sqrt{6}$, b = $\sqrt{7}-\sqrt{6}$ Then ab = 1 and a + b = $2\sqrt{7}$ From $(a+b)^2=a^2+2ab+b^2$ We have: $(\sqrt{7}+\sqrt{6})^2=a^2=(2\sqrt{7})^2-2-b^2=26-b^2$ $\because$ $0<b^2<1$ $\therefore$ the smallest integer is 26.
12	5	Put r = $\sqrt{5}-2$ $r^2+2r+2\sqrt{5}=(\sqrt{5}-2)^2+2(\sqrt{5}-2)+2\sqrt{5}=5$
13	$\frac{\sqrt{2}}{2}$ (or $\frac{1}{\sqrt{2}}$)	 Because M and N are mid-points of AB and CD respectively and △DMC and △ANB are isosceles triangles, MN is perpendicular to DC and AB.



Let M, N be midpoints of AB and CD respectively. If P, Q are other points on these segments, ΔPQM and ΔQNM are right-angled triangles. Thus, PQ > QM and QM > MN, MN is the shortest distance between the two edges.

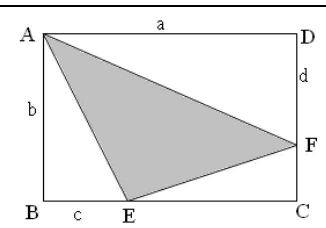
Consider ΔBMN, which is a right-angled triangle,

$$MN^2 = BN^2 - BM^2 = \left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2$$

14 | **216**

Length	Breadth	Height	No. of rectangular blocks
3	3	3	1
3	3	2	6
3	3	1	9
3	2	2	12
3	2	1	36
3	1	1	27
2	2	2	8
2	2	1	36
2	1	1	54
1	1	1	27

15 | **15.5 (cm^2)**



Let AD = a, AB = b, BE = c and DF = d.

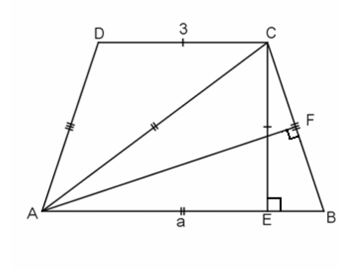
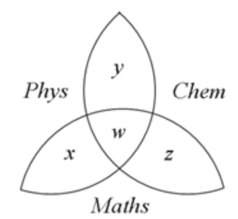
$\therefore$ ab = 35, ad = 14, bc = 10 and $cd = \left(\frac{10}{b}\right)\left(\frac{14}{a}\right)$=4

Then the area of ΔCEF

$= \frac{(a-c)(b-d)}{2}$

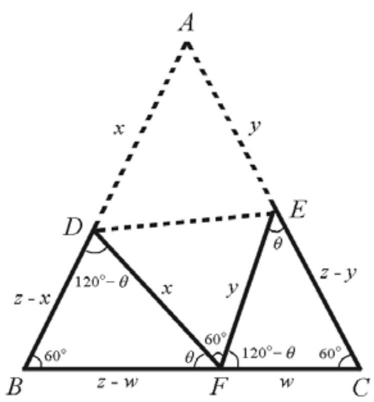
$= \frac{ab - ad - cb + cd}{2} = 7.5$

$\therefore$ The area of ΔAEF = 35 – 5 – 7 – 7.5 = 15.5 (cm^2)

16	5	E and F are chosen such that AF and CE are perpendicular to CB and AB respectively. $\because \Delta AFB \sim \Delta CEB$ $\therefore \frac{FB}{EB} = \frac{AB}{CB} \Rightarrow \frac{\frac{1}{2}CB}{EB} = \frac{AB}{CB} \Rightarrow \frac{\frac{1}{2}CB}{\frac{1}{2}(a-3)} = \frac{a}{CB}$ $\Rightarrow (CB)^2 = a(a-3)$ ΔBCE is a right-angled triangle, i.e.  $(CB)^2 = 3^2 + \left(\frac{a-3}{2}\right)^2$ $\therefore \quad a^2 - 2a - 15 = 0$ $a = 5 \text{ or } a = -3 (\text{rejected})$
17	5	**Answer**:  $x + y + w = 12$, $y + z + w = 15$, $z + x + w = 18$, $x + y + z + w = 20$. Hence, $2 \times 20 + w = 12 + 15 + 18$. w = number of students studying all three subjects $= (12 + 15 + 18) - 2 \times 20 = 5$.
18	A wins 4 times or 1 time.	The corresponding outcomes are: (1) (*A*, *B*) = (✋,✊), (✋,✊), (✋,✌), (✌,✊), (✌,✋), (✊,✌), or (2) (*A*, *B*) = (✋,✌), (✋,✌), (✋,✊), (✌,✊), (✌,✊), (✊,✋)

乙部 Part B

No.	Answer	Explanation
19	1504	Note that : 1. $2k + 1 = (k+1)^2 - k^2 \quad (k = 1,2,3...)$ 2. $4k = (k+1)^2 - (k-1)^2 \quad (k = 2,3,4...)$ 3. $a^2 - b^2 = (a+b)(a-b)$ which is an odd number or a multiple of 4 larger than 4. Hence, the 'wise integers' are odd numbers larger than 1 and multiples of 4 larger than 4. $\therefore$ The 'wise integers' are 3,5,7,8,9,11,12,13,15,16,17,19,20,…. i.e. there are $(\frac{3}{4} \times 2008 - 2 =) 1504$ wise integers.

20	48 cm^2	Join *AF* [This is the crucial construction line. With it, the computation is no more hard!] Note that $\frac{x}{2}=\frac{x+y+6}{6}, \frac{y}{6}=\frac{x+y+2}{10}$. Hence we have $4x-2y=12$, $6x-4y=-12$. Solving the equations, we obtain $x=18$, $y=30$. Hence, $x+y=48$.
21	140	Let $x = AD$, $y = AE$, $z = BC$ and $w = FC$. Let $\theta = \angle BFD$.  Note that ΔFBD is similar to ΔECF. Therefore, we have $$\frac{EC}{FB}=\frac{CF}{BD}=\frac{FE}{DF}$$ $$\frac{z-y}{z-w}=\frac{w}{z-x}=\frac{y}{x}.$$ Hence, $w=\frac{(z-x)y}{x}$ and $\frac{z-y}{z-\frac{(z-x)y}{x}}=\frac{y}{x}$. Solving z in terms of x and y, we have $z=\frac{xy(x+y)}{x^2-xy+y^2}$. Since $x = 91$ and $y = 65$, we have $BC = z = 140$.

顧問：蕭文強教授(香港大學)、吳端偉博士(香港大學)

擬題委員會：

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