香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2006 – 2007 Solution

甲部 Part A

1. $\boxed{2007x-1}$

   $x(y+2007)-xy-1=2007x-1$.

2. $\boxed{a<-b<b<-a}$

   Since $a+b<0$, then $a<-b$ and $b<-a$.
   Since $a<b$ and $ab<0$, then $a<0$, $b>0$ and $-b<b$.
   $\therefore\ a<-b<b<-a$.

3. $\boxed{x+y=-\frac{1}{2}}$

   $$\begin{cases} x+y=xy & (1) \\ xy=\dfrac{x}{y} & (2) \end{cases}$$

   From (2), $y^2=1$ then $y=\pm 1$ since $x\neq 0$
   When y = 1, $x+1=x$ (rejected)
   When y = –1, $x-1=x$ and $x=\frac{1}{2}$
   $\therefore$ The two numbers are $\frac{1}{2}$ and –1, then $x+y=\frac{1}{2}-1=-\frac{1}{2}$

4. $\boxed{n=6}$

   Since $n-5\geq 0$, $\therefore$ n should be ≥ 5.
   when $n=5$, $7\sqrt{n-5}-n = 7\sqrt{5-5}-5 = -5$ (rejected)
   when $n=6$, $7\sqrt{n-5}-n = 7\sqrt{6-5}-6 = 1$ is accepted.
   $\therefore$ $n=6$

5. $\boxed{54}$

   The sum of the 5 integers = $77\times 5=385$
   sum of the 2 smallest integers = $385-97-97-87=104$
   the smallest integer = $97-47=50$
   $\therefore$ the second smallest integer = $104-50=54$

6. $\boxed{x=4 \text{ or } 5}$

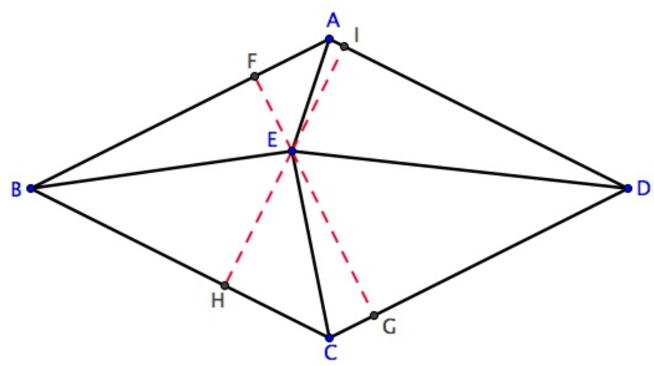
   case(i): $5x-16=3x-8$
   $2x=8$
   $x=4$
   case(ii): $5x-16=3x-6$
   $2x=10$
   $x=5$

7. $\boxed{168^\circ}$

   Since ΔAPB is equilateral, $AP=PB=AB$, $\therefore \angle APB=60^\circ$.
   Since ABCDE is regular, $\angle BAE=108^\circ$.
   $\therefore \angle PAE=108^\circ-60^\circ=48^\circ$
   Since $AE=AB=BC=AP=PB$,
   $\therefore \Delta APE$ and ΔBPC are isosceles,
   $\therefore \angle APE=\angle BPC=(180^\circ-48^\circ)/2=66^\circ$.
   $\therefore \angle CPE=360^\circ-66^\circ-60^\circ-66^\circ=168^\circ$

8. $\boxed{14\text{ cm}^2}$

As shown in the figure, we consider the altitudes of the 4 triangles drawn from the common vertex E. They should lie on two straight lines through E and the sums FE + EG and IE + EH should be equal. We can therefore prove that the sum of areas of each pair of opposite triangles should be half of that of the rhombus.



$\therefore$ The area of $\Delta DAE = 22+10-18=14\text{ cm}^2$

9. $\boxed{a=2}$, $\boxed{b=11}$

Since $a^2+b^2=125$ and b is odd, then a must be even.

$\therefore a=2$ since 2 is the only even prime number.

$\therefore b=\sqrt{125-2^2}=11$.

10. $\boxed{500}$

Since $10=2\times 5$ and number of multiples of 2 is more than multiples of 5, then we may count the number of factor 5 in

$2007!=2007\times 2006\times 2005\times \ldots \times 3\times 2\times 1$ to get the answer.

$2007!$ has $\left[\frac{2007}{5}\right]+\left[\frac{2007}{25}\right]+\left[\frac{2007}{125}\right]+\left[\frac{2007}{625}\right]$ factors of 5.

$\therefore 2007!$ has $401+80+16+3=500$ trailing zeros.

11. $\boxed{\frac{3}{2}}$

$$\begin{cases} 2a^2+2007a+3=0 \ldots\ldots(1) \\ 3b^2+2007b+2=0 \ldots\ldots(2) \end{cases}$$

$$\begin{cases} 2a+\frac{3}{a}=-2007 \\ 3b+\frac{2}{b}=-2007 \end{cases}$$

$$(1)-(2): \quad 2a+\frac{3}{a}-3b-\frac{2}{b}=0$$

$$(2a-3b)\left(1-\frac{1}{ab}\right)=0$$

$$\frac{a}{b}=\frac{3}{2} \quad \text{or} \quad ab=1\,(\text{rejected})$$

Alternative method:

Put $x=\frac{1}{b}$ into $2x^2+2007x+3=0$,

$$LHS=2\left(\frac{1}{b}\right)^2+2007\left(\frac{1}{b}\right)+3=\frac{1}{b^2}(2+2007b+3b^2)=0=RHS$$

$\therefore a$ and $\dfrac{1}{b}$ are roots of $2x^2+2007x+3=0$,

$\therefore$ product of roots = $\dfrac{a}{b}=\dfrac{3}{2}$

12. $\boxed{5\ cm}$

$(2r)^2=(16-2r)^2+(18-2r)^2$

$r^2-34r+145=0$

$(r-5)(r-29)=0$

$r=5 \quad or \quad r=29(rejected)$

13. $\boxed{\dfrac{26}{3}\text{cm}^2}$

Let DE = x cm and AE = EC = $(6-x)$cm

In ΔAD'E,

$$4^2+x^2=(6-x)^2$$
$$16+x^2=36-12x+x^2$$
$$12x=20$$
$$x=\frac{5}{3}$$

$\therefore$ Area of ΔAD'E = $\dfrac{1}{2}\left(\dfrac{5}{3}\right)(4)$ = $\dfrac{10}{3}cm^2$

Since $\Delta AD'E \cong \Delta ABF$

$\therefore$ Area of $ABFE$ = Area of $AD'EF$ = Area of $CBFE$

$$= \frac{1}{2}\times 4\times 6 = 12\text{ cm}^2$$

Area of ΔAEF = $12-\dfrac{10}{3}=\dfrac{26}{3}\text{cm}^2$

14. $\boxed{3, 5, 7}$

$abc=7(a+b+c)$

since a, b and c are prime, so one of a, b, c should be 7.

w.l.g. put a = 7, then $bc=7+b+c$

$\therefore\ c=\dfrac{b+7}{b-1}$

when b = 3, then c = 5.

when b = 5, then c = 3.

15. $\boxed{\text{the number of rooms = 7}}$

$\boxed{\text{the total number of students = 58}}$

$$9(x-1)\le 4x+30<9x$$

$9x-9\le 4x+30$ and $4x+30<9x$

$x\le 7.8$ and $x>6$

$\therefore x = 7$

$\therefore$ the number of rooms = 7

$\therefore$ the total number of students = $4\times 7+30=58$

16. $\boxed{4\text{ cm}^2}$

The pyramid *E-ABCD* consists of 5 faces: a square and 2 pairs of congruent right triangles. If the area of a square face of the cube is 1 sq unit, then areas of triangle *ABE* and *BCE* are $1/2$ and $\sqrt{2}/2$ sq. units respectively. Surface area of pyramid *E-ABCD* = $1+2(1/2)+2(\sqrt{2}/2)=2+\sqrt{2}$.

The pyramid *E-ACD* consists of 2 pairs of congruent right triangular faces, the areas of which are $1/2$ (for triangles *ACD* and *ADE*) and $\sqrt{2}/2$ (for triangle *CDE* and *CAE*). Its surface area is therefore $1+\sqrt{2}$.

The ratio of the surface areas of pyramid *E-ABCD* to pyramid *E-ACD*

$$=\frac{2+\sqrt{2}}{1+\sqrt{2}}=\frac{\sqrt{2}\left(\sqrt{2}+1\right)}{\left(1+\sqrt{2}\right)}=\sqrt{2}.$$

Surface area of pyramid *E-ABCD* = $\sqrt{8}\times\sqrt{2}=4$.

17. $\boxed{PD=11}$

Draw *EF*//*AD*//*BC*,

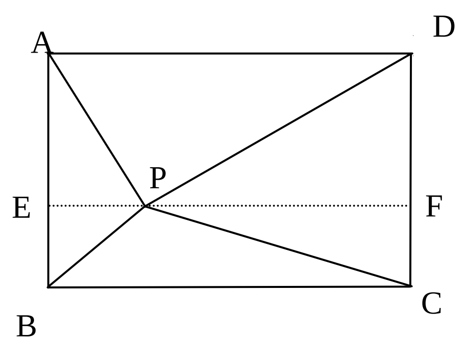
$$PA^2-AE^2=PB^2-BE^2$$
$$PD^2-DF^2=PC^2-CF^2$$

Hence $PA^2-PD^2=PB^2-PC^2$

Or $PD^2=PA^2+PC^2-PB^2$

$$=5^2+10^2-2^2=11^2$$

Hence *PD* = 11.



18. $\boxed{75, 84, 93, 102, 111, 120, 129, 138, 147, 156}$

$S = (10a + b) + (10c + d) + (10e + f)$

$= 10(a + c + e) + (b + d + f)$

$\equiv a + b + c + d + e + f \pmod 9$

$\equiv 1 + 2 + 3 + 4 + 5 + 6 \equiv 3 \pmod 9$.

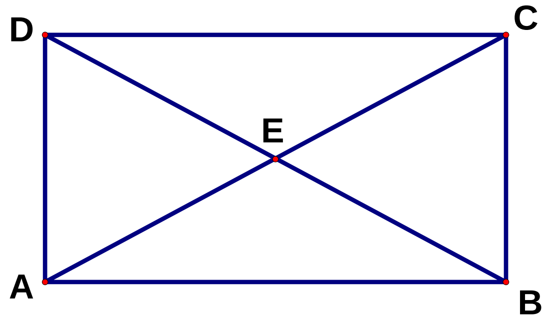
The smallest value for S is 75 = 14 + 25 + 36, and the largest value for S is 156 = 63 + 52 + 41.

The possible candidates are the ten numbers from 75 to 156 that are congruent to 3 modulo 9. It is not hard to check that all ten numbers are attainable sums.

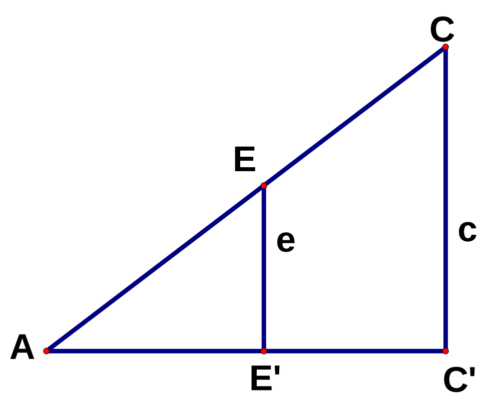
[There are 120 possible groups of three such numbers and. So it is too time-consuming to calculate all attainable sums in this way.]

乙部 Part B

19 According to the property of rectangle, E is the mid-point of both line segments AC and DB.



From the side view,

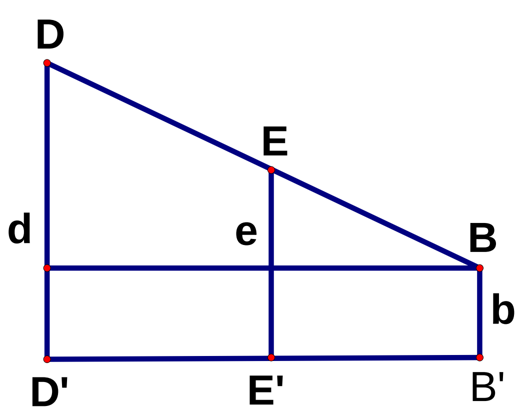


∵ AE = EC and EE' // CC'

∴ AE' = E'C' (intercept theorem)

and EE' = CC' (mid-point theorem)

i.e. $e = \frac{c}{2}$



∵ DE = EB and EE' // DD' // BB'

∴ D'E' = E'B' (intercept theorem)

and d – b = 2(e – b) (mid-point theorem)

Substituting $e = \frac{c}{2}$ into the above equation

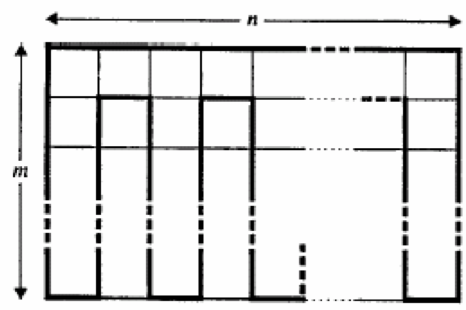
we have c = b + d

20. 5

We need to know in which ‘segment’ the 2007th digit falls. In the string of ‘segments’ 1, 2, 3, . . . , 9, there are 1 + 2 + 3 +· · ·+ 9 = 45 digits. In the string that follows, with ‘segments’ 10, 11, . . ., 44, there are 2 × (10 + 11 + · · · + 44) = 1890 digits. The string with ‘segment’ 45 that follows has 2 × 45 = 90 digits. Since 45 + 1890 < 2007 < 45 + 1890 + 90, we see that the 2007th digits falls in the ‘segment’ 45. Because 2007 – 45 – 1890 = 72 is even, the digit in the 2007th position is 5. (To estimate in which ‘segment’ the 2007th digit falls, we look at $45 + 2 \times (n + 10)(n - 10 + 1)/2 \approx 2007$, which gives $n^2 \approx 2052$, so $n \approx 45$.)

21. (i) $m = n = 1$, or
(ii) $m > 1$ and n is even, or
(iii) $n > 1$ and m is even.
For any other m, n, it is not possible to do so.

Remark:
If $m = n = 1$, then it is trivially possible.
If($m = 1$ and $n > 1$) or ($m > 1$ and $n = 1$), then it is clearly impossible.
If m, $n > 1$ and one of m, n is even, then it is possible as shown in the figure.
(In the figure, n is taken to be even.)



Suppose $m > 1$, $n > 1$, m and n are both odd. Denote the number of moves to the left, right, up, and down by l, r, u and d respectively. If it is possible to visit each house just once and return to the starting point, then $l + r + u + d = mn$, which is odd. But, to return to the starting point, $l = r$ and $u = d$, so this is impossible.