香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2005 – 2006

Suggested Solutions(English Version Only)

1. Ans. : 7:12

Note : One way to interpret the problem is to think of a 24-hour clock with two hands both starting at 0:00. One hand (representing the slow clock) moves with (angular) speed $(1-\frac{30}{60})\times\frac{2\pi}{24}$ and the other hand (representing the fast clock) moves with (angular) speed $(1+\frac{20}{60})\times\frac{2\pi}{24}$. We want to find the time t (in hour-minute) after which the difference in angle traversed by the two hands is π, that is, $(\frac{80}{60}\times\frac{2\pi}{24}-\frac{30}{60}\times\frac{2\pi}{24})\times t=\pi$, or t = 72/5. Hence, after 14 hours and 24 minutes (the true time), the slow clock shows 7:12 (a.m.) while the fast clock shows 7:12 (p.m.). This is the first instance after 0:00 when both clocks look the same. (The actual time is 2:24 p.m.)

Another way to solve the problem is let t be the number of hours passed when two clocks look exactly the same for the first time. Then |t/2-4/3t| has to be equal to 12n for some integer n. If n=1, then t=72/5. One can then verify that t=72/5 is actually a solution to the problem.

2. Ans : There are many possible solutions

e.g
1 4 2 3
3 2 4 1
4 1 3 2
2 3 1 4

Note : Basically there are only two solutions, namely,

1 3 4 2	1 4 2 3
4 2 1 3	3 2 4 1
2 4 3 1	4 1 3 2
3 1 2 4	2 3 1 4 .

All other solutions are obtained from either one by
(i) permuting 1,2,3,4, (ii) rotation, (iii) reflection, or combination thereof. To get the two, first put on one diagonal 1,2,3,4 in that order, then deduce the rest.

3. Ans : 2007

$$\frac{\left(2^2+4^2+...+2006^2\right)-\left(1^2+3^2+5^2+...+2005^2\right)}{\left(2+4+...+2006\right)-(1+3+5+...+2005)}$$

$$=\frac{(2006^2-1^2)+(2004^2-3^2)+...+(2^2-2005^2)}{\left(2+4+...+2006\right)-(1+3+5+...+2005)}$$

$$=\frac{(2006+1)(2006-1)+(2004+3)(2004-3)+...+(2+2005)(2-2005)}{\left(2+4+...+2006\right)-(1+3+5+...+2005)}$$

$$=\frac{2007[(2006-1)+(2004-3)+...+(2-2005)]}{\left(2+4+...+2006\right)-(1+3+5+...+2005)}$$

$= 2007$

Or $\dfrac{\left(2^2+4^2+...+2006^2\right)-\left(1^2+3^2+5^2+...+2005^2\right)}{\left(2+4+...+2006\right)-(1+3+5+...+2005)}$

$=\dfrac{(2^2-1^2)+(4^2-3^2)+\cdots+(2004^2-2003^2)+(2006^2-2005^2)}{(2-1)+(4-3)+\cdots+(2004-2003)+(2006-2005)}$

$=\dfrac{(2-1)(2+1)+(4-3)(4+3)+\cdots+(2004-2003)(2004+2003)+(2006-2005)(2006+2005)}{1003}$

$=\dfrac{3+7+\cdots+4007+4011}{1003}$

$=\dfrac{(3+4011)(1003)\div 2}{1003}$

$=2007$

4. Ans : 330

$1 + 2 + 3 + 4 + x_5 + (x_5+1) + (x_5 + 2)+ (x_5 + 3) + (x_5 + 4) + (x_5 + 5) \le 2006$

$2006 - 1 - 2 - 3 - 4 - 1 - 2 - 3 - 4 - 5 > 6x_5$

$x_5 \le (2006 - 25) / 6 = 330.1$,

The greatest positive integer satisfying the above inequality is 330.

5. Ans. : 2006

Consider

$9 \mid \overline{62xy427}$:

$9 \mid 6 + 2 + x + y + 4 + 2 + 7$, $9 \mid x + y + 3$, $x + y = 6$ or 15

$11 \mid \overline{62xy427}$:

$11 \mid (6 + x + 4 + 7) - (2 + y + 2)$, $11 \mid 13 + x - y$, $x - y = -2$,

$x = 2$, $y = 4$

$500x + 250y + 6 = 500(2) + 250(4) + 6 = 2006$

6. Ans. : 982

Consider $n^3 + 3n + 2006$

$= n^2 (n + 3) - 3n^2 + 3n + 2006$

$= n^2 (n + 3) - 3n (n + 3) + 12n + 2006$

$= n^2 (n + 3) - 3n(n+3) + 12(n+3) + 1970$

If $(n +3) \mid n^2 (n + 3) - 3n (n + 3) + 12n + 2006$,

$(n +3) \mid 1970$

$1970 = 2(5)(197)$

$n + 3 = 197$ or $n + 3 = 394$ or $n + 3 = 985$

$n = 194$ or $n = 391$ or $n = 982$

Therefore the greatest possible value of n is 982

7. Ans. : $\sqrt[3]{5}$

$\because$ $[x]$ and 4 are integers, $\therefore$ x^3 is an integer

Thus $x = \sqrt[3]{N}$ and N should be an integer too.

Now, find N such that $N - [\sqrt[3]{N}] = 4$

When $N = 2, 3, 4, 5, 6, 7$, $\sqrt[3]{N} = 1$

$\therefore$ $N = 5$ and $x = \sqrt[3]{5}$

8. Ans. : (10,1)

Explanation: Reflect A in the perpendicular bisector of BC.

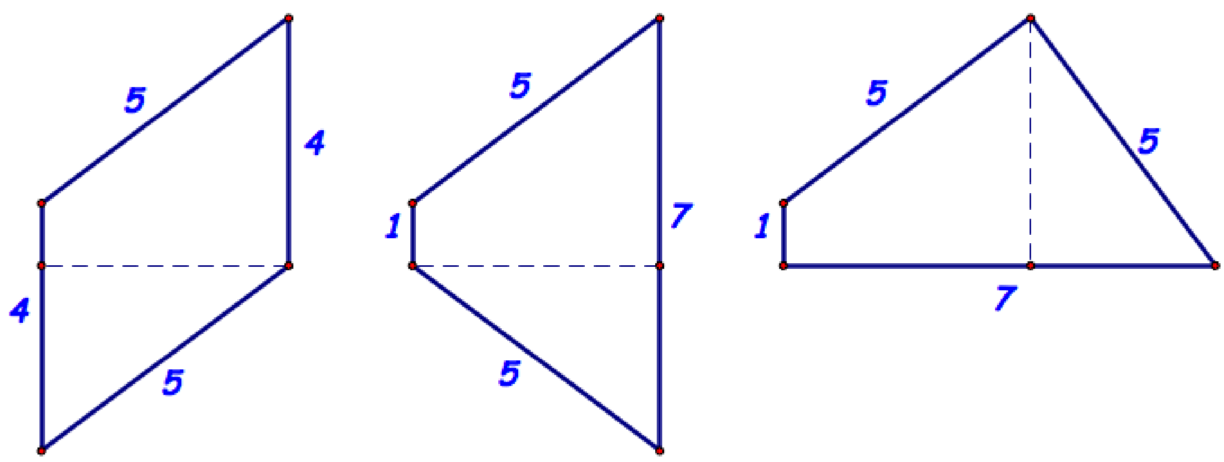
9. Ans. : 18 cm

1^{st} quadrilateral: 18 cm

2^{nd} quadrilateral: 18 cm

3^{rd} quadrilateral: 18 cm

Explanation: These are the quadrilaterals formed.



10. Ans. : –4

Let the numbers be x and y. Since

$x + y = xy = 2$,

$8 = (x + y)^3 = x^3 + y^3 + 3xy(x + y) = x^3 + y^3 + 3(2)(2)$

$= x^3 + y^3 + 12$

Hence $x^3 + y^3 = -4$.

11. Ans. : 8 km/h

Let c be the speed of the current and s the speed of the ferry in still water. Then the downstream and upstream speeds are m + s and m – s respectively. After the ferry doubles its still water speed, the downstream and upstream speeds are 2m + s and 2m – s respectively. Since the distance is 60 km and the time is distance ÷ speed, we have

$$\frac{60}{m+s} = \frac{60}{m-s} - 5 \quad \text{and} \quad \frac{60}{2m+s} = \frac{60}{2m-s} - 1$$

Simplifying the equations to get

$5m^2 - 5s^2 = 120s$ and $4m^2 - s^2 = 120s$

Substituting to get m = 2s, we get s = 8.

12. Ans. : 41

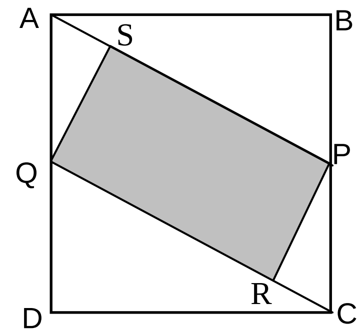
The equation is equivalent to $y = \frac{3(167 - x)}{4}$. For y to be a positive integer, (167 – x) must be a positive multiple of 4. This is true for 41 positive integers x = 4k + 3, k = 0, 1, 2, ..., 40.

13. Ans. : $22 + 8\sqrt{6}$

Let $x = \sqrt{4+\sqrt{4+\sqrt{4}}}$. Since $\sqrt{4} = 2$, $x = \sqrt{4+\sqrt{6}}$, $x^2 = 4+\sqrt{6}$,

and $x^4 = (x^2)^2 = 16 + 8\sqrt{6} + 6 = 22 + 8\sqrt{6}$

14. Ans. : 40%



Let AB = $2x$. Then BP = QD = x,

AP = QC = $\sqrt{5}x$ [Pythagoras's Theorem]

By considering the two similar triangles,

e.g. Consider △ AQS and △ABP

$$\frac{QS}{2x}=\frac{AS}{x}=\frac{x}{\sqrt{5}x},$$

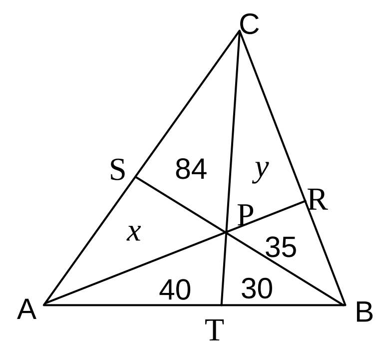
QS = RP = $\frac{2}{\sqrt{5}}x$, AS = RC = $\frac{1}{\sqrt{5}}x$,

Therefore the area of the no shaded part = $(x)(2x)+(\frac{2}{\sqrt{5}}x)(\frac{1}{\sqrt{5}}x)=2.4x^2$

The area of the shaded part = $1.6x^2$

The proportion of the shaded area to the square ABCD is 40%.

15 Ans. : 315



∵ Area of ΔAPB : Area of ΔRPB = 70 : 35, AP:PR = 2 : 1 and

Area of ΔACP : Area of ΔRCP = 2: 1

∴ $84 + x = 2y$

Similarly

Area of ΔBCP : Area of ΔBTP = Area of ΔACP: Area of ΔATP

$$\frac{y+35}{30}=\frac{x+84}{40}$$

$4(y+35) = 3(2y)$

$y = 70$, $x = 56$

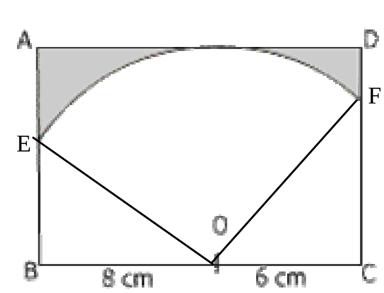
The area of ΔABC = 84 + 35 + 30 + 40 + 70 + 56 = 315

16. Ans. : $\frac{44\pi}{3}$ cm

It can be shown that the triangle has to travel around the square 3 times before its vertices are again in their initial positions. The point P undergoes 24 moves, 8 of them $\frac{1}{3}$ of one revolution, 8 of them $\frac{7}{12}$ of one revolution, and 8 of them traverse no path. Thus the length of the path traveled is

$$8\times 2\pi\times\frac{1}{3}+8\times 2\pi\times\frac{7}{12}=\frac{44\pi}{3}cm.$$

17. Ans. : 92 - 25π (cm)



Join OE and OF.

In ΔOBE,

OB : OE = 8 : 10 = 4 : 5

and BE : OB : OE = 3 : 4 : 5

∴BE = 6cm

In ΔOCF,

OC : OF = 6 : 10 = 3 : 5

and OC : CF : OF = 3 : 4 : 5

$\therefore$ CF = 8 cm

$\Theta\ \Delta OBE \cong \Delta OCF$

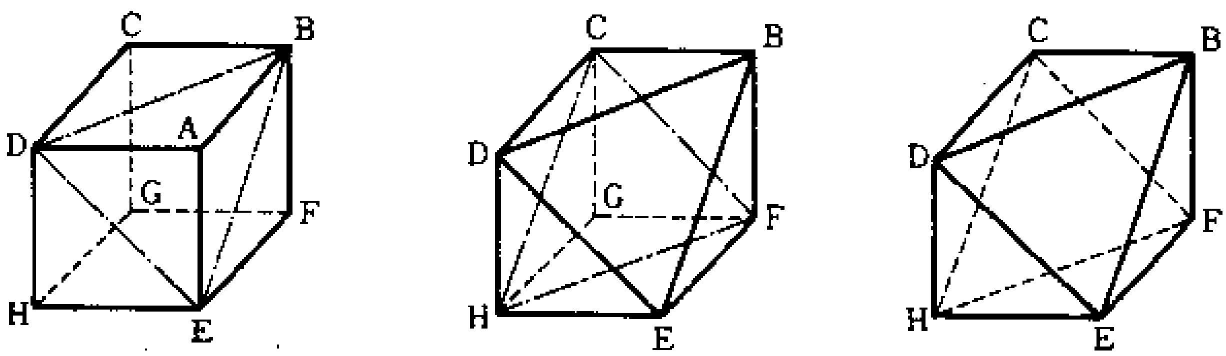
$\therefore \angle EOF = 90^{\circ}$

Area of sector OEF = 25π

The area of the shaded region = (14)(10) - 25π - 48 = 92 - 25π (cm)

18. Ans. : $5\frac{1}{3}cm^3$

The required solid is formed from cutting two pyramids from a cube of side 2cm. Consider a cube of side 2cm. Two pyramids are cut off along the plain BDE and CFH.



Therefore the volume of the required solid is:

$2^3 - 2(2\times 2 \div 3) = 5\frac{1}{3}cm^3$

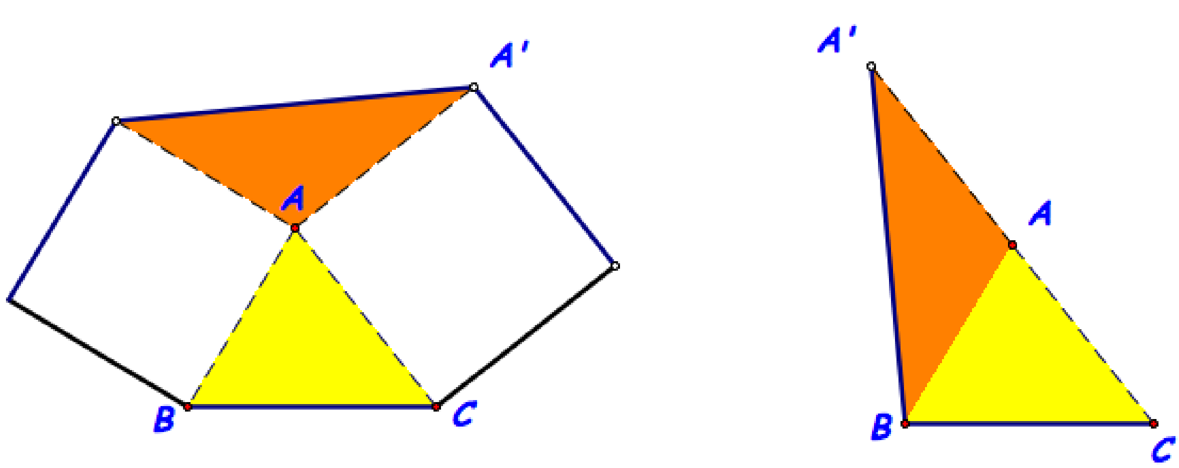
Part B

19. Ans. : 2,2,3,7,8.

Note : Let the five positive integers be a,b,c,d,e. Let S be their sum. Then the five sums obtained by taking four at a time are S-a, S-b, S-c, S-d, S-e. These five sums add up to 4*S. Four of the five are mutually distinct, meaning that two of them are equal, say x. Hence, we have 14+15+19+20+x=4*S, or x=4*S-68=4*(S-17). There is only one among the four values that is divisible by 4, so x must be 20 and S=22. These gives all five positive integers : 22-20=2, 22-20=2, 22-19=3, 22-15=7, 22-14=8.

20. Ans. : 4.5

Explanation: When the upper triangle is rotated about A by 90 degrees, both triangles share the same vertex B and have equal bases AC and AA'. Therefore the 2 triangles have equal areas.



21. a. Ans. : 1/2

Note : The possible seating arrangements are (A B C), (B A C), (C A B), (C B A). Since A is equally likely to be seated in any seat,

P(ABC) = P(CBA) = P(BAC or CAB) = 1/3.

When B finds his place occupied, he is equally likely to be seated in the other two seats,

P(BAC) = P(CAB) =1/6

P(ABC or BAC) = 1/3 + 1/6 = 1/2

Or

the probability of (ABC) is 1/3.

The probability of (CBA) is 1/3

When A chooses the second seat, there are 2 choices left to B.

The probability of (BAC) is 1/3 x 1/2

The probability of (CAB) is 1/3 x 1/2

Therefore the required probability is 1/3 x 1/2 + 1/3 = 1/2

b. Ans. : 1/2

Note : The answer is still 1/2. In fact, this is true for N men with any N not less than 2. Suffices to illustrate with the case of four men A,B,C,D.

(1) If A sat in the first seat, then the only possible seating is (A B C D).

(2) If A sat in the second seat, then the only possible seating arrangements are (B A C D), (C A B D), (D A B C), (D A C B). Note that B is forced to behave like drunk while there are three seats for B,C,D to choose from.

(3) If A sat in the third seat, then the only possible seating arrangements are (C B A D), (D B A C). Note that C is forced to behave like drunk while there are two seats for C,D to choose from.

(4) If A sat in the fourth seat, then the only possible seating is (D B C A).

Hence, the chance of D sitting in the correct seat is 1 in (1), 1/2 in both (2) and (3), and 0 in (4). Altogether, the chance of the fourth man (D) sitting in the correct seat is 1/2. The same line of argument goes for any N men as long as N is 2 or larger.

P(seating 4 men with the first drunk) = ¼ (1) + ¼(1/2) + ¼(1/2) + ¼ (0)=1/2

Similarly,

P(seating 5 men with the first drunk)

$$= \frac{1}{5}(1) + \frac{1}{5}(\frac{1}{2}) + \frac{1}{5}(\frac{1}{2}) + \frac{1}{5}(\frac{1}{2}) + \frac{1}{5}(0) = \frac{1}{2}$$

P(seating n men with the first drunk)

$$= \frac{1}{n}(1) + \underbrace{\frac{1}{n}(\frac{1}{2}) + \frac{1}{n}(\frac{1}{2}) + \cdots + \frac{1}{n}(\frac{1}{2})}_{(n-2)terms} + \frac{1}{n}(0) = \frac{1}{2}$$