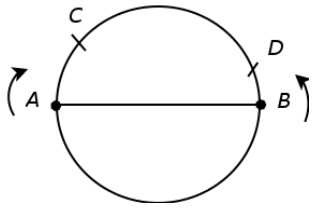
香港青少年數學精英選拔賽
The Hong Kong Mathematical High Achievers Selection Contest
2004 – 2005

建議題解大綱
Suggested Sketch of Solutions

SOLUTIONS

1. ANSWER: 2 minutes later



EXPLANATION: From start to the first encounter Keung and Wai together cover half of the circle. From the first encounter to the second encounter Keung and Wai together cover a full circle. If the first encounter occurs at 1 minute after start, the second encounter would occur at 2 minutes later (after the first encounter).

[Alternatively one can do some calculation. Suppose Keung and Wai start from *A*, *B* with uniform speeds u, v respectively (with the appropriate unit omitted). Let they meet the first time at *C* after 1 minute and the second time at *D* after *t* minutes later. Then $u+v=\pi R$ and $ut+vt=2\pi R$ where R is the radius of the circular track. Hence $t=2$.]

答案: 2 分鐘之後
解釋: 從開始至第一次相遇，小強和小偉合共走了半個圓。從第一次相遇至第二次相遇，小強和小偉合共走了一個圓。如果從開始至第一次相遇用了 1 分鐘，從第一次相遇至第二次相遇便用了 2 分鐘。[也可以做一點計算，設小強和小偉的速度分別是u, v(單位省略了)，得到$u+v=\pi R$，R是圓的半徑。如果第一次相遇後再過 *t* 分鐘兩人又再相遇，便有$ut+vt=2\pi R$，即是說$t=2$。]

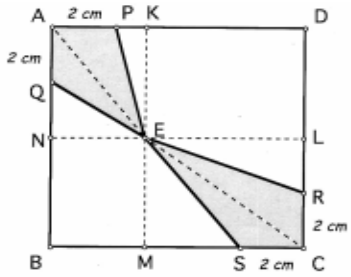
2. ANSWER: 64 cm^2

EXPLANATION: The quadrilaterals can be considered as 4 triangles with same base length 2 cm and heights EK, EL, EM and EN, which are perpendiculars from E to the sides of the square. Combined area of the quadrilaterals

$$=\frac{1}{2}\cdot 2(EK+EL+EM+EN)$$
$$=KM+LN$$
$$=2\times AB$$

AB = 8 cm and the area of the square = 64 cm^2

答案: 64 cm^2
解釋: 把兩個四邊形分爲 4 個三角形。每個三角形的底邊均是 2 cm, 高分別是由 E 至正方形各邊的垂線 EK, EL, EM 和 EN。兩個四邊形的面積和是:



$$=\frac{1}{2}\cdot 2(EK+EL+EM+EN)$$
$$=KM+LN$$
$$=2\times AB$$

所以, AB = 8 cm 而正方形的面積是 64 cm^2。

3. ANSWER: 74°
EXPLANATION:
148° = ∠GEH + ∠GHE (ext. ∠ of Δ)
so ∠GHE = 88° and hence ∠BHF = 88° (Vert. Opp. ∠s) and thus ∠CFE = 148°(ext. ∠ of Δ).
Now, as ΔCDF and ΔEDF are congruent, x = 74°.

答案 : 74°
解釋 :
148° = ∠GEH + ∠GHE (Δ 外角)
∴ ∠GHE = 88° 及 ∠BHF = 88° (對頂角)。
由此 ∠CFE = 148° (Δ 外角)。
由於 ΔCDF 及 ΔEDF 爲全等，x = 74°.

4. ANSWER: 25

EXPLANATION:

By Pythagoras Theorem, $5^2 = BC^2 - CE^2 = (CD + AB)^2 - (CD - AB - 1)^2$; thus
$25 = CD^2 + 2AB \cdot CD + AB^2 - CD^2 + 2AB \cdot CD + 2CD - AB^2 - 2AB - 1$. Hence,
$4AB \cdot CD + 2CD - 2AB - 1 = 25 = (2AB + 1)(2CD - 1)$.

答案 : 25

解釋:

利用畢氏定理，$5^2 = BC^2 - CE^2 = (CD + AB)^2 - (CD - AB - 1)^2$ ；所以
$25 = CD^2 + 2AB \cdot CD + AB^2 - CD^2 + 2AB \cdot CD + 2CD - AB^2 - 2AB - 1$.
由此，$4AB \cdot CD + 2CD - 2AB - 1 = 25 = (2AB + 1)(2CD - 1)$.

5. ANSWER: $\frac{1}{2}L$
EXPLANATION: Let the length of the six edges be a, a', b, b', c, c' where a, a' are that of opposite edges, likewise with b, b' and c, c'. Hence $(a+b+c)+(a'+b'+c') = L$. But $a+b+c = a'+b+c' = a'+b'+c = a+b'+c' = p$ where p is the perimeter of each face, so $(a+b+c)+(a'+b'+c') = 2p$. Therefore, $p = \frac{1}{2}L$.

答案: $\frac{1}{2}L$
解釋: 設四面體六條則棱的長是a, b, c和a', b', c'，其中a和a'是對邊的長，類似的b和b'，c和c'。所以$(a+b+c)+(a'+b'+c') = L$。但我們也知道$a+b+c = a'+b+c' = a'+b'+c = a+b'+c' = p$，$p$是每面三角形的周界。所以$(a+b+c)+(a'+b'+c') = 2p$，因此，$p = \frac{1}{2}L$。

6. ANSWER: The two integers are 21 and 24.
EXPLANATION: Let the two integers be a and b.
Since $168 = 2^3\times3\times7$, so $a = 2^x\times3^y\times7^z$ and $b = 2^d\times3^e\times7^f$
where x, y, z, d, e, and f are integers such that $0\le x, d \le 3$, $0\le y, e \le 1$ and $0\le z, f \le 1$.
Now as $a + b = 45$, we have $2^x\times3^y\times7^z + 2^d\times3^e\times7^f = 45$.
From this, we know that only one of a and b is even, without loss of generality, we have $x = 3$ and $d = 0$. Similarly, as 45 is divisible by 3, $y = e = 1$. By then, it is clear that $z = 0$ and $f = 1$.
Therefore, the two integers are $2^3\times3 = 24$ and $2^3\times7 = 21$.

答案：該兩個數是 21 和 24

解釋：設該兩個數是 a 和 b。
因爲 $168 = 2^3\times3\times7$，所以 $a = 2^x\times3^y\times7^z$ 及 $b = 2^d\times3^e\times7^f$，
x、y、z、d、e 和 f 是整數並且 $0\le x, d \le 3$, $0\le y, e \le 1$ 和 $0\le z, f \le 1$.
現在 $a + b = 45$，我們可得 $2^x\times3^y\times7^z + 2^d\times3^e\times7^f = 45$.
由此，我們得知 a 和 b 其中只有一個是偶數，不失一般性的情況下，
可得 $x = 3$ 和 $d = 0$。相似地，因 45 可被 3 整除，$y = e = 1$。
因此，明顯地 $z = 0$ 和 $f = 1$。
所以，該兩個數是 $2^3\times3 = 24$ 和 $2^3\times7 = 21$。

7. ANSWER: – 10

EXPLANATION: Since $(n-1)a_n = a_1 + a_2 + \cdots a_{n-1}$, then
$a_{n+1} = \dfrac{a_1 + \cdots a_n}{n} = \dfrac{(n-1)a_n + a_n}{n} = a_n$. Thus, $a_3 = \dfrac{20 + a_2}{2}$. Hence, $a_2 = -10$.

答案: – 10

解釋: 由於 $(n-1)a_n = a_1 + a_2 + \cdots a_{n-1}$，因此 $a_{n+1} = \dfrac{a_1 + \cdots a_n}{n} = \dfrac{(n-1)a_n + a_n}{n} = a_n$。所以，
$a_3 = \dfrac{20 + a_2}{2}$。由此， $a_2 = -10$。

8. ANSWER: $\dfrac{20}{7}$

EXPLANATION: Extend *CD* to the last lattice point at the corner *F* and form two similar triangles $\Delta AEC \sim \Delta BEF$. $AB = \sqrt{3^2 + 4^2} = 5$ and $AE : EB = 4 : 3$. Thus,
$AE = 5\cdot\dfrac{4}{7} = \dfrac{20}{7}$.

答案: $\dfrac{20}{7}$

解釋: 將 *CD* 伸延到最後一個格點至角 *F* ，從而形成兩個相似三角形 $\Delta AEC \sim \Delta BEF$。
$AB = \sqrt{3^2 + 4^2} = 5$ 及 $AE : EB = 4 : 3$。 所以， $AE = 5\cdot\dfrac{4}{7} = \dfrac{20}{7}$。

9. ANSWER: 4

EXPLANATION: The sum of the first five numbers is 35, and the sum of the last five numbers is 50. The sum of all nine numbers is 81. Thus the fifth number must be $50+35-81=4$.

答案: 4

解釋: 前五個數之和是 35，後五個數之和是 50，全部九個數之和是 81。所以第五個數必須是 $50+35-81=4$。

10. ANSWER: 177

EXPLANATION: Since a^b must be divisible by 7 and 7^2 but not any higher powers, the largest b is 2. Hence, $5^4 7^2 = 30625^1 = 175^2$ and the smallest $a+b$ is 177.

答案: 177

解釋: 由於a^b能被 7 及 7^2而不是 7 的更高次方所整除，故 b 的最大值是 2。所 以，$5^4 7^2 = 30625^1 = 175^2$及$a+b$的最小值是 177。

11. ANSWER:13

EXPLANATION: Let the number of this group of workmen be x.

Then xy + 12 = 8(x – 1) + 7, so that xy + 12 = 8x – 1 and hence 13 = (8 – y)x. The only possible case is 8 – y = 1 and x = 13.

答案: 13

解釋: 設工人的數目是 x。即 xy + 12 = 8(x – 1) + 7，得出 xy + 12 = 8x – 1 及 13 = (8 – y)x。故唯一可能的情況是 8 – y = 1 及 x = 13。

12. ANSWER: $\left(\frac{2004}{2005}\right)^2$

EXPLANATION: Since AC – BC = 2005 – 2004 = 1

$\mathbf{AC^2 - BC^2}$ = (AC + BC)(AC – BC)

= (2005 + 2004)(2005 – 2004) = 2005 + 2004 = AB^2

We have $AC^2 = AB^2 + BC^2$, i.e., $\angle B = 90^o$.

$\sin \mathbf{A} = \frac{\mathbf{2004}}{\mathbf{2005}}$ and $\cos \mathbf{C} = \frac{\mathbf{2004}}{\mathbf{2005}}$, so sinA×cosC = $\left(\frac{2004}{2005}\right)^2$

答案: $\left(\frac{2004}{2005}\right)^2$

解釋: 由於 AC – BC = 2005 – 2004 = 1

$\mathbf{AC^2 - BC^2} = (AC + BC)(AC - BC)$

$= (2005 + 2004)(2005 - 2004) = 2005 + 2004 = AB^2$

得出 $AC^2 = AB^2 + BC^2$ 即 $\angle B = 90^o$.

$\sin \mathbf{A} = \frac{\mathbf{2004}}{\mathbf{2005}}$ 及 $\cos \mathbf{C} = \frac{\mathbf{2004}}{\mathbf{2005}}$ ，所以 $\sin A \times \cos C = \left(\frac{2004}{2005}\right)^2$

13. ANSWER: The Church clock is 2 minutes slower than the home clock.
The school hall clock is 1 minutes slower than the home clock.
EXPLANATION: Comparing the times shown at the home clock, the time Ming used in traveling was 66 minutes, i.e., he used 33 minutes to walk to school. Therefore, the school hall clock is 1 minutes slower than the home clock. On the other hand, by comparing the times shown in the Church clock, the time used in traveling from Church to school was 11 minutes, so that the Church clock is 2 minutes slower than the home clock.

答案: 教會的大鐘比家裡的大鐘慢 2 分鐘

學校禮堂的大鐘比家裡的大鐘慢 1 分鐘

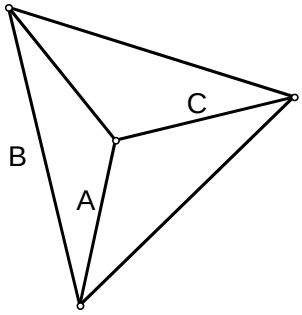
解釋：與家裡的大鐘比較, 小明來回共用去 66 分鐘。(他上學用去 33 分鐘。) 所以，禮堂的鐘比家裡的大鐘慢了 1 分鐘。另一方面, 由教堂到學校需要 11 分鐘, 所以教堂的鐘比家裡的大鐘慢了 2 分鐘。

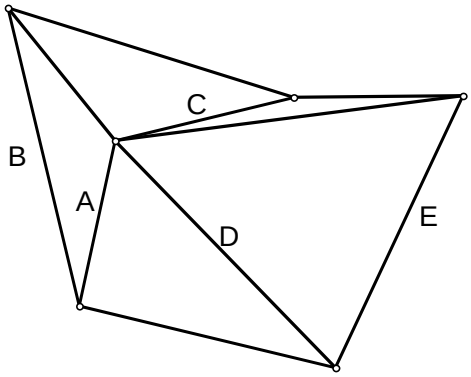
14. ANSWER: 7
EXPLANATION: From 1 to 9 there are 9 digits. From 10 to 99 there are $2 \times 90 = 180$ digits. From 100 to 999 there are $3 \times 900 = 2700$ digits. Since 2005 lies between $9 + 180 = 189$ and $9 + 180 + 2700 = 2889$, we look for the 2005^{th} digit among the block formed by 3-digit numbers. Note that $2005 = 9 + 180 + 3 \times 605 + 1$, so the 2005^{th} digit is the one next to 704, which is 7.

答案: 7
解釋: 從 1 至 9 有 9 個數字，從 10 至 99 有 $2 \times 90 = 180$ 個數字，從 100 至 999 有 $3 \times 900 = 2700$ 個數字。由於 2005 大於 $9 + 180 = 189$ 卻小於 $9 + 180 + 2700 = 2889$，那個數的第 2005 個數字出現在三位數組成的一段。注意到 $2005 = 9 + 180 + 3 \times 605 + 1$，便知道那個數的第 2005 個數字落在 704 的旁邊，即是 7。

15. Answer: 36 cm^3

Net 1 will be folded to make a regular tetrahedron. The solid made from net 2 is equivalent to 3 regular tetrahedrons sharing two common faces, as shown in the following.

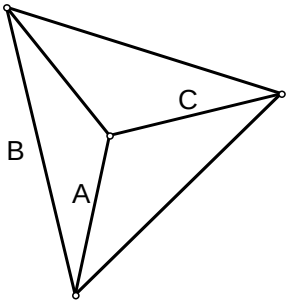


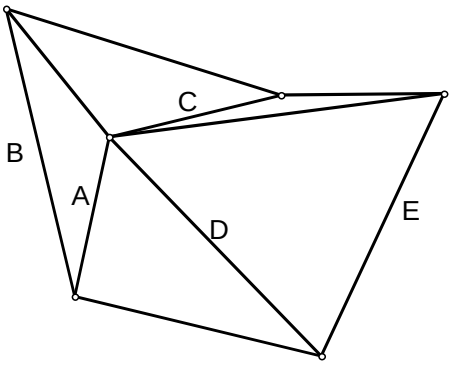


Further information and a dynamic model of this solid are available at
http://mathworld.wolfram.com/Boat.html

答案：36 cm^3

摺紙圖樣 1 可以摺成正四面體。而摺紙圖樣 2 可以摺成一相當於由三個正四面體合拼而成，當中每個正四面體各有一個面爲共同面，如下圖所示：





其他資料與及該立體的動態模型可參考以下網頁：
http://mathworld.wolfram.com/Boat.html

16. ANSWER : 9

EXPLANATION: As each digit of A is not greater than 9, so

$a \le 9 \times 2005 = 18045$

so a is at most a five-digit integer.

By then, $b \le 5 \times 9 = 45$ and hence $c \le 2 \times 9 = 18$.

Now as $9 \mid A$, we have also $9 \mid a$, $9 \mid b$, and $9 \mid c$. But as $c \le 18$, c can only be 9 or 18.

If $c = 18$, $b = 99$, which is impossible, so $c = 9$.

答案 ：9

解釋：因爲 A 的每一個數值都不大於 9，所以 $a \le 9 \times 2005 = 18045$，a 最多是一個位數。

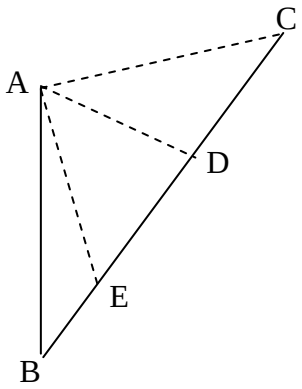
另外，$b \le 5 \times 9 = 45$ 及 $c \le 2 \times 9 = 18$。

因爲 $9 \mid A$，所以 $9 \mid a$, $9 \mid b$, 及 $9 \mid c$。但是 $c \le 18$，c 只可以是 9 或 18。

若 $c = 18$, $b = 99$，這是沒有可能的。所以 $c = 9$。

17. ANSWER : $4\sqrt{15}$ hours

EXPLANATION: The question can be represented by the following diagram :

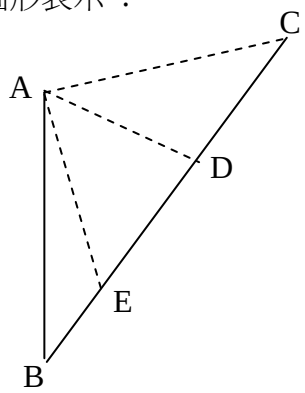


Hong Kong is at the point A and the typhoon is at the point B. We first find the shortest distance between Hong Kong and the path of the movement of the typhoon, which is AD where ∠ADB = 90°. Now as ∠ABD = 30°, and AB = 220 km, we have AD = 110 km. Thus wind strength at Hong Kong when the typhoon is at D is 12 – 110/20 = (level)6.5. Therefore, the Observatory needs to hoist typhoon signals.

On the other hand, from 12 – x/20 = 4, we have x = 160, thus when the typhoon is 160 km away from Hong Kong, a typhoon signal needs to be hoisted. AE and AC are drawn such that AE = AC = 160 km. DE = $\sqrt{AE^2 - AD^2} = 30\sqrt{15}$ and hence CE = $60\sqrt{15}$ km. Therefore, we have the time when the typhoon traveling from E to C is $60\sqrt{15} / 15 = 4\sqrt{15}$ hours.

答案 : $4\sqrt{15}$ 小時

解釋：這問題可由以下圖形表示 :



香港在 A 點及颱風在 B 點。我們首先尋找香港與颱風的路徑的最短距離，即 AD， 其中 ∠ADB = 90°。現在 ∠ABD = 30°，及 AB = 220 km，得到 AD = 110 km。因此當颱風在 D 時香港的風力強度是 12 – 110/20 = 6.5 級。所以天文台需要懸掛颱風訊號。

另一方面，由 12 – x/20 = 4，得到 x = 160，因此當颱風離開香港 160 km 時，颱風訊號需要懸掛。繪畫 AE 及 AC 使到 AE = AC = 160 km。DE = $\sqrt{AE^2 - AD^2} = 30\sqrt{15}$ 及 CE = $60\sqrt{15}$ km。所以，當颱風由 E 點移動至 C 點的時間是 $60\sqrt{15} / 15 = 4\sqrt{15}$ 小時。

18. EXPLANATION: Flatten out the four faces of the tetrahedron into a network on the plane (see Figure 1), which, by what is given, actually forms a triangle with intercepts joining the midpoints of each pair of corresponding sides (see Figure 2). The proof can now be completed, using the Midpoint Intercept Theorem in plane geometry.

解釋: 把四面體的各面攤開成爲平面上的一個多邊形(見 Figure 1)。從給定條件可知這個多邊形原來是一個三角形，而且其餘的線段都是把每邊的中點相應地聯結起來(見 Figure 2)。利用平面幾何的中點截段定理，可知四面體的對棱同長。

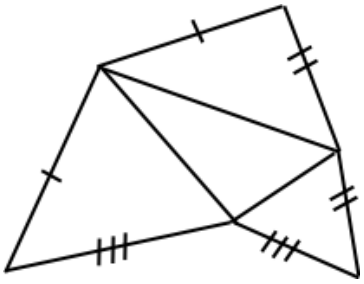


Figure 1

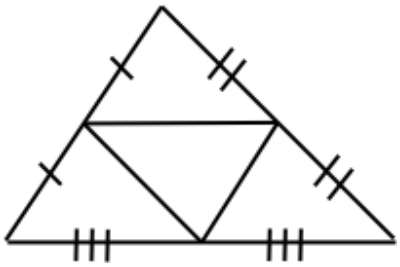


Figure 2