香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2003 – 2004

建議題解大綱

Suggested Sketch of Solutions

1. $\boxed{1+\sqrt{3}+\sqrt{5}}$

把方程 $\sqrt{7+2(1+\sqrt{3})(1+\sqrt{5})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$ 的左右雙方同時平方之後可得

$x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}=9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$。由比較同類項的係數及解對應的方程就可得出該答案。

Square both sides of $\sqrt{7+2(1+\sqrt{3})(1+\sqrt{5})}=\sqrt{x}+\sqrt{y}+\sqrt{z}$ to get

$x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}=9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15}$. Compare coefficients and solving respective equations to get the answer.

2. $\boxed{2014}$

由 $x=\dfrac{\sqrt{11}-1}{3}$ 可得 $3x+1=\sqrt{11}$。將左右雙方平方可得 $9x^2+6x-10=0$。

From $x=\dfrac{\sqrt{11}-1}{3}$ it is obtained that $3x+1=\sqrt{11}$. Square both sides to get $9x^2+6x-10=0$.

3. $\boxed{6}$

這裡有 6 個答案：(1, 3), (3, 1), (2, 2), (1, 2), (2, 1), (1, 1)。

There are six solutions: (1, 3), (3, 1), (2, 2), (1, 2), (2, 1), (1, 1).

4. $\boxed{0.79}$

$b = 17$
$a = 17 - 2 = 15$
$c = 17 + 2 = 19$
$a \div c = 15 \div 19 = 0.79$(準確至二位小數)

$b = 17$
$a = 17 - 2 = 15$
$c = 17 + 2 = 19$
$a \div c = 15 \div 19 = 0.79(2d.p.)$

5. $\boxed{2}$

方法一：

$$
\begin{aligned}
\left(\sqrt{32} - \sqrt{18}\right)^2 &= 32 - 2\sqrt{32}\sqrt{18} + 18 \\
&= 50 - 2\sqrt{32 \times 18} \\
&= 50 - 2\sqrt{64 \times 9} \\
&= 50 - 2 \times 8 \times 3 \\
&= 2
\end{aligned}
$$

(Method 1)

$$
\begin{aligned}
\left(\sqrt{32} - \sqrt{18}\right)^2 &= 32 - 2\sqrt{32}\sqrt{18} + 18 \\
&= 50 - 2\sqrt{32 \times 18} \\
&= 50 - 2\sqrt{64 \times 9} \\
&= 50 - 2 \times 8 \times 3 \\
&= 2
\end{aligned}
$$

方法二：
由比
ABCD 的面積：PQRC 的面積
= 32 : 18
= 16 : 9
可知
$DC : RC = 4 : 3$ 及 $DC : DR = 4 : 1$
所以
$RSTD$ 的面積是 $ABCD$ 面積的 1/16。

(Method 2)

From the ratio

Area of *ABCD* : Area of *PQRC*

= 32 : 18

= 16 : 9

we know

DC : *RC* = 4 : 3 and *DC* : *DR* = 4 : 1

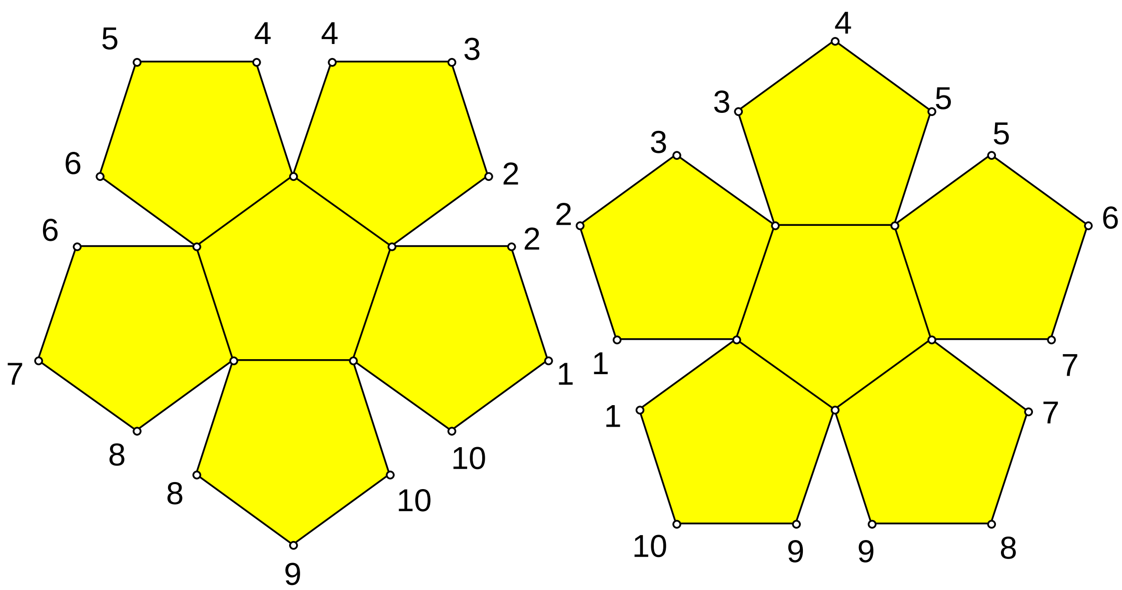
therefore

Area of *RSTD* is 1/16 of Area of *ABCD*

6. D

各頂點根據以下的數字重疊。

The vertices are renamed below to show the matching.



7. $\left(\frac{5}{9}\right)^{10}$

每一代餘下的空白面積都是前一代的空白面積的 5/9。

At each stage, 5/9 of the white portion in the previous stage remains unshaded.

8. $\boxed{115200}$

$$\begin{aligned}2^3+4^3+6^3+\ldots+30^3 &= 2^3\cdot 1^3+2^3\cdot 2^3+2^3\cdot 3^3+\ldots+2^3\cdot 15^3\\ &= 2^3(1^3+2^3+3^3+\ldots+15^3)\\ &= 115200\end{aligned}$$

9. $\boxed{35}$

設甲地至乙地的距離爲 s。設輪船在靜止的水上行駛時的速度爲每日 v，而水的速度爲每日 u。

首先，建立 $\begin{cases} s=5v+5u \\ s=7v-7u \end{cases}$

然後，把 v 消去，得 $\dfrac{s}{u}=35$。所以，木排需要 35 日從甲地順流漂到乙地。

Let the distance between A and B be s. Let the speed of the ship in still water be v per day and the speed of the water flow be u per day.

Then $\begin{cases} s=5v+5u \\ s=7v-7u \end{cases}$

By eliminating v, we get $\dfrac{s}{u}=35$. Therefore, it takes 35 days for wood floating from the A to B.

10. $\boxed{\sqrt{32}=4\sqrt{2}}$

在 AB 加上圓心 O。這樣 AO = OB = 6。連接 OC，而 OC = 6。OD = OB – DB，即 OD = 6 – 4 = 2。所以 $CD=\sqrt{6^2-2^2}=\sqrt{32}=4\sqrt{2}$。

Add O to be the center of circle on AB. Then AO = OB = 6. Join OC, OC = 6. OD = OB – DB, i.e. OD = 6 – 4 = 2. Then $CD=\sqrt{6^2-2^2}=\sqrt{32}=4\sqrt{2}$.

11. $\boxed{\dfrac{25}{36}}$

從方程式 $\dfrac{x}{1.1}+\dfrac{y}{1.25}=\dfrac{x+y}{1.2}$ 開始，其中 x 爲男選民的數目而 y 爲女選民的數目，計算後得出 $\dfrac{x}{y}=\dfrac{11}{25}$，及 $\dfrac{x+y}{y}=\dfrac{36}{25}$。所以 $\dfrac{y}{x+y}=\dfrac{25}{36}$。

Begin with the equation $\frac{x}{1.1}+\frac{y}{1.25}=\frac{x+y}{1.2}$, where x is the number of male voters and y is the number of female voters, it can be computed that $\frac{x}{y}=\frac{11}{25}$, and $\frac{x+y}{y}=\frac{36}{25}$. Thus, $\frac{y}{x+y}=\frac{25}{36}$.

12. $\boxed{\frac{4007}{2004}}$

因爲 $a_{n+1}=\frac{1}{2004}+a_n$ ，從這可以觀察到 $a_n=\frac{n-1}{2004}+1$ 。

Since $a_{n+1}=\frac{1}{2004}+a_n$, it can be observed that $a_n=\frac{n-1}{2004}+1$ in general.

13. $\boxed{18}$

整數 8, 8, 8, 8, 10, 10, 10, 10, 10, 18 滿足要求的條件，所以最大整數的可能值最小為 18。如最大數值為 19，則數列變成 9, _ , _ , _ , 10, 10, _ , _, _, 19。

平均數爲 10 即十個整數的和爲 100，六個未填寫的數值的和是 52 及平均值是 $8\frac{2}{3}$ ，因此最少一個未填寫的數值要少於 9。形成矛盾。

The integers 8, 8, 8, 8, 10, 10, 10, 10, 10, 18 satisfy the requirements. Therefore, the answer is at least 18. If the largest number were 19, then the collection becomes 9, _ , _ , _ , 10, 10, _ , _, _, 19. The mean is 10 means that the sum of all integers is 100. The sum of the six missing values is 52, and the mean of the six missing values is $8\frac{2}{3}$. Thus, at least one of the missing values is less than 9, which is a contradiction.

14. $\boxed{324}$

由計算可得 $a_8=8a_1+13a_2$ 及 $a_7=200=5a_1+8a_2$。因此 8 爲 a_1 的倍數，a_1 的可能數值是 32, 24, 16 及 8，首三個數值令 $a_2<a_1$ 。

因此，$a_1=8$ 及 $a_2=20$ 。

It can be calculated that $a_8=8a_1+13a_2$ and $a_7=200=5a_1+8a_2$. Thus, 8 is a factor of a_1. The only possible values of a_1 are 32, 24, 16 and 8. The first three values will lead to $a_2<a_1$. Thus, $a_1=8$ and $a_2=20$.

15. $\boxed{30^o}$

解釋：將圖形沿 AB 反射。將C，D與影像 C’ 及 D’ 分別地連線。ACBC’ 爲一正方形，

CC’ = AB

DD’ = CC’ = AB

已知 AB = BD, DD’ = BD = BD’.

BDD’ 爲一等邊三角形，

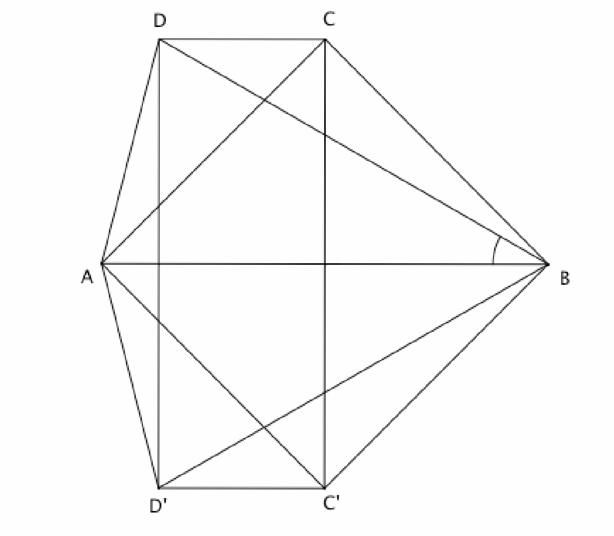
$\angle$DBD’ = 60^o 及 $\angle$ABD = 30^o

Explanation: Reflect the figure about AB. Join C, D to their images C’ and D’. ACBC’ is a square; CC’ = AB.

DD’ = CC’ = AB.

Given AB = BD, DD’ = BD = BD’. BDD’ is an equilateral triangle.

$\angle$DBD’ = 60^o and $\angle$ABD = 30^o.



16. 解釋：注意到 ΔABX 與 ΔDEX 有相同的面積。(因爲 $ABCD$ 的面積 與 $BCDE$ 的面積均爲六角形面積的一半。)

同理，ΔBCY 與 ΔEFY 有相同的面積；而 ΔCDZ 與 ΔAFZ 亦有相同的面積。 由此，

$ab=(a'+a'')(b'+b''), b''c''=(b+b')(c+c'), a''c=(a+a')(c'+c'')$ 。

將第一式乘 c 得 $abc=a'b'c+a'b''c+a''b'c+a''b''c$ 。

將第二式乘 a 得 $ab''c''=abc+abc'+ab'c+ab'c'$ 。

將第三式乘 b'' 得 $a''b''c=ab''c'+ab''c''+a'b''c'+a'b''c''$ 。

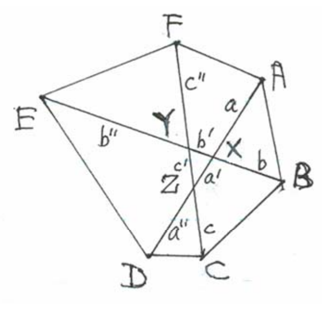
將各式相加化簡後得 $0=a'b'c+a'b''c+a''b'c+abc'+ab'c+ab'c'+ab''c'+a'b''c'+a'b''c''$

由此，得 $a'b'c=0$ 。但由於 c 不等於 0，所以 $a'=0$ 或 $b'=0$ 。無論是那個情況，都得出 AD、BE、CF 相交於一點。

(另外，也可將該三式相乘而得出

$abc\,a''b''c''=(a+a')(b+b')(c+c')(a''+a')(b''+b')(c''+c')>abc\,a''b''c''$ 若 a'、b'、c' 其中一個值不等於 0。

因此，$a'=b'=c'=0$ 而得 AD、BE、CF 相交於一點。)



Explanation: Note that ΔABX and ΔDEX have the same area, because together with $BCDX$ each occupies a half of the hexagon. Similarly, ΔBCY and ΔEFY have the same area, ΔCDZ and ΔAFZ have the same area. Hence

$ab=(a'+a'')(b'+b''), b''c''=(b+b')(c+c'), a''c=(a+a')(c'+c'')$.

Multiply c to the first throughout to get $abc=a'b'c+a'b''c+a''b'c+a''b''c$.

Multiply a to the second throughout to get $ab''c''=abc+abc'+ab'c+ab'c'$.

Multiply b'' to the third throughout to get $a''b''c=ab''c'+ab''c''+a'b''c'+a'b''c''$.

Add up the three to obtain $0=a'b'c+a'b''c+a''b'c+abc'+ab'c+ab'c'+ab''c'+a'b''c'+a'b''c''$.

In particular, $a'b'c = 0$, but c cannot be zero, hence $a' = 0$ or $b' = 0$. In either case, that means *AD*, *BE*, *CF* are concurrent.
(Alternatively, we can multiply the three to obtain
$abc\,a''b''c'' = (a + a')(b + b')(c + c')(a'' + a')(b'' + b')(c'' + c') > abc\,a''b''c''$ if one of a', b', c' is positive.
Therefore, $a' = b' = c' = 0$ and *AD*, *BE*, *CF* are concurrent.

17. $\boxed{\frac{1}{3}}$

解釋：我們可假設 *ABCDHEFG* 爲一單位立方體。 由此可直接求出三角錐體 *ACHF* 的邊長爲 $\sqrt{2}$，而體積則是 $\frac{1}{3}$。

Explanation: We may assume *ABCDHEFG* is a unit cube. One way is to compute directly the volume of the regular tetrahedron *ACHF* with all sides having length $\sqrt{2}$. It turns out to be equal to $\frac{1}{3}$.

另一較快捷的方法是將正立方體看成由四個與 *CHGF* 相同的四面體，加上 *ACHF* 所組成。該四個相同的四面體可組成一個高爲 1，底爲 $\sqrt{2}$ 的金字塔。因此，

$ACHF$ 的體積 $= 1 - (\frac{1}{3})(1)(2) = \frac{1}{3}$。

A much quicker way is to observe that four pieces, each identical to *CHGF* (a tetrahedron of some special shape), together with *ACHF*, make up the cube. These four pieces combine together to form a pyramid with height 1 and square base of sides having length $\sqrt{2}$. Hence

Volume of $ACHF = 1 - (\frac{1}{3})(1)(2) = \frac{1}{3}$.

18. $\boxed{334, 336, 338}$

由於 $4 \times 6 \times 8$ 的個位是 2 而 $0 \times 2 \times 4$, $2 \times 4 \times 6$, $6 \times 8 \times 0$, $8 \times 0 \times 2$ 的個位均不是 2，所以第一個偶數的個位是 2。因爲 $3^3 = 27$ 而 $4^3 = 64$，所以第一個偶數的百位是 3。最後，因爲 33^3 的首兩位是 35 而 34^3 的首兩位是 39，所以第一個偶數的十位是 3。

The last digital of the first even integer must be 4, because $4 \times 6 \times 8$ ends with 2, while none of $0 \times 2 \times 4$, $2 \times 4 \times 6$, $6 \times 8 \times 0$, $8 \times 0 \times 2$ ends with 2. The first digit must be 3, because $3^3 = 27$, while $4^3 = 64$. The second digit must be 3, because 33^3 begins with 35, while 34^3 begins with 39.