香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2002-2003

建議題解 Suggested Solutions

甲部 Part A

1. Ans. 20

金頌會於下列日子購買紙張 ：

Kam-chung will purchase on the following days :

$$\left[\frac{20}{3}\right]=6、\left[\frac{40}{3}\right]=13、\left[\frac{60}{3}\right]=20、\ \cdots$$

思恒則會於下列日子購買紙張 ：

Sze-hang will purchase on the following days :

$$\left[\frac{20}{5}\right]=4、\left[\frac{40}{5}\right]=8、\left[\frac{60}{5}\right]=12、\left[\frac{80}{5}\right]=16、\left[\frac{100}{5}\right]=20\ \ldots,$$

其中 $[a]$ 代表 a 的整數部分。因此，他們將於第 20 日一同去購買紙張。

where $[a]$ denotes the integral part of a. Hence, they will go to purchase together at the 20^{th} day.

2. Ans. $a=2, b=1, c=5, d=4, e=3$

將各式加起，可得 $2(a+b+c+d+e)=30$，由此， $a+b+c+d+e=15$。而 $(a+b)+(c+d)=3+9=12$，所以 $e=3$，利用代入法可得 $a=2, b=1, c=5, d=4$。

Sum up the expressions, we get $2(a+b+c+d+e)=30$, and hence $a+b+c+d+e=15$. Since $(a+b)+(c+d)=3+9=12$, we get $e=3$. By successive substitution, $a=2, b=1, c=5, d=4$.

3. Ans. 2003125

$2003125^2-(2003124)(2003126)=1$，所以答案是 2003125。

$2003125^2-(2003124)(2003126)=1$, so the answer is 2003125.

4. Ans. 5

$\overbrace{2+0+0+3+\cdots+2+0+0+3}^{10組}=50$， 因爲 50 被 9 時的餘數是 5，所以所求答案是 5。

$\overbrace{2+0+0+3+\cdots+2+0+0+3}^{10sets}=50$. The remainder of $50\div9$ is 5. The answer is 5.

5. Ans. 4

考慮個位數的演變 ：

Consider only the unit digits :

n	1	2	3	4	5	…	42
18^n 的個位數 The unit digit of 18^n	8	4	2	6	8		

$18^{42} = 18^{10\times4+2}$ ：18^{42} 的個位數 = 18^2 的個位數 = 4。

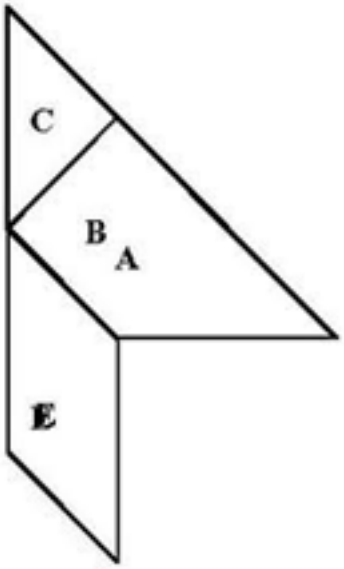
$18^{42} = 18^{10\times4+2}$; the unit digit of 18^{42} = the unit digit of 18^2 = 4.

6. Ans. 22

$a \# b = 50 - 2 \times (a + b)$。 所以，$6 \# 8 = 22$。

$a \# b = 50 - 2 \times (a + b)$. So, $6 \# 8 = 22$.

7. Ans. B,E ; A,C,E



8. Ans. (8,4), (6,2), (5,7), (3,5)

由於 $DE = BC$，因此應是對應邊。而 DE 垂直於 BC，因此 DEF 應是由 ABC 旋轉 90 度(順時針或逆時針) 或先旋轉 90 度後再反射而得。

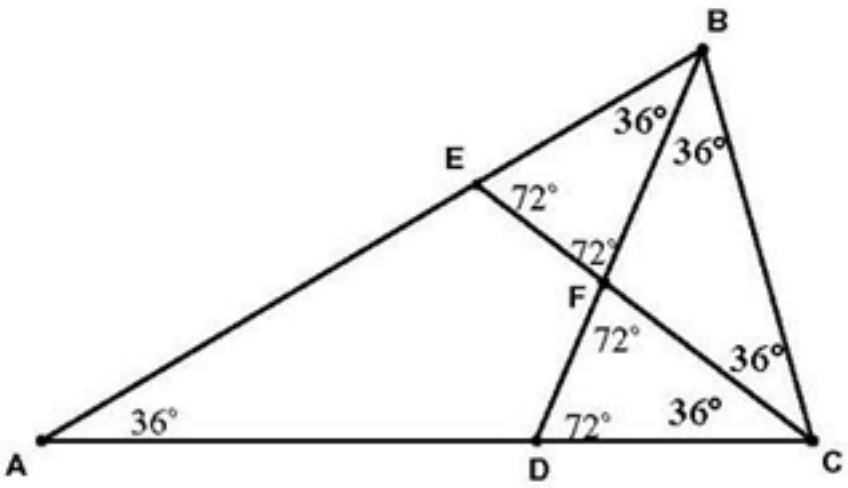
DE is equal to BC in length and they should be corresponding sides. DE is perpendicular to BC. DEF should be an image of ABC when turned through a right angle (clockwise or anti-clockwise) with or without flipping.

9. Ans. 8

其餘的角度可以被求得。圖中的所有三角形都是等腰三角形：

The remaining angles can be found. All triangles in the figure are isosceles:

$\Delta ABC, \Delta FBC, \Delta BEF, \Delta CDF, \Delta BDC, \Delta CEB, \Delta ADB, \Delta AEC$.



10. Ans. $7+\frac{13\pi}{3}$

對於這個問題，我們只需計算所有加在該立方体上(在 1cm³ 以內)使它成爲一個新形狀的所有不同形狀的體積。這些額外的小塊爲：

- 每面上的長方体。[6×1]
- 每邊上的四分一的圓柱体。$[12\times(1/4)\times\pi\times1^2\times1=3\pi]$
- 每頂點上的八分一的圓球体。$[8\times(1/8)\times(4/3)\pi\times1^3=4\pi/3]$

所以總体積 = $1+6+3\pi+\frac{4\pi}{3}=7+\frac{13\pi}{3}$。

This is a problem where we just have to keep count of the various shapes which are added onto the cube (within 1 m³) to form the new shape. These extra bits are:

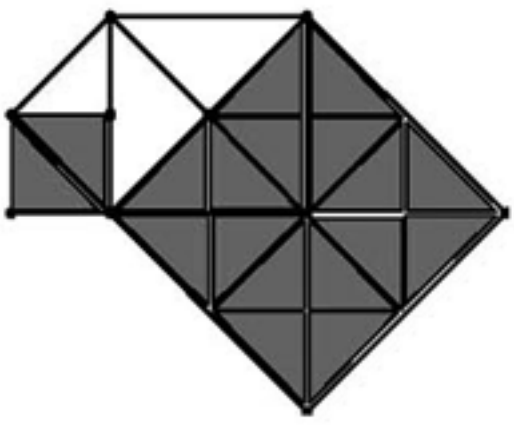
- a cube on each face [6×1]
- a quarter cylinder for each edge $[12\times(1/4)\times\pi\times1^2\times1=3\pi]$
- an one-eighth sphere on each vertex $[8\times(1/8)\times(4/3)\pi\times1^3=4\pi/3]$

Thus the total volume $=1+6+3\pi+\frac{4\pi}{3}=7+\frac{13\pi}{3}$.

11. Ans. 28

每個正方形的對角線是下一個較大正方形的邊，它的面積剛好是下一個較大正方形的面積的一半。我們可以把最大的正方形切成十六個相等的等腰三角形，而每一個都是最小正方形的一半。

A diagonal of each square is a side of the next larger square, and its area is half of that of the next square. We may also divide the largest square into 16 right isosceles triangles, each equal to half of the smallest square.



12. Ans. 3600000001

考慮所有由 0 到 100000000 的整數。它們可以配對成 0+99999999，1+99999998，2+99999997 等等。這便成了一共 50000000 對數。每對數的位數合計爲 $9\times8=72$。所以，所有位數的總數值爲 $50000000\times72+1=3600000001$。

Consider all integers from 0 to 100000000. They can be paired up into 0 + 99999999, 1 + 99999998, 2 + 99999997, etc. There are totally 50000000 pairs of numbers. The digits of all these pairs add up to $9 \times 8 = 72$. Therefore, sum of all digits = $50000000 \times 72 + 1 = 3600000001$.

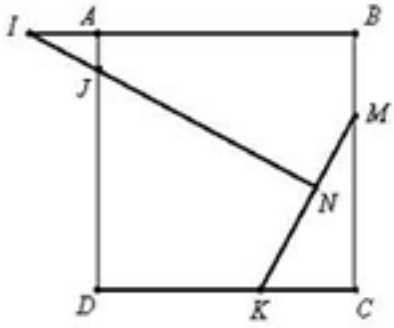
13. Ans. 50

剩餘的兩個三角形有相同的面積(x 平方單位)，比較三角形的底邊，可得$\frac{x}{8}=\frac{18}{x}$。由此，$x=12$。 ∴梯形的面積$=8+18+12+12=50$。

Note that the other two triangles have the same area (x square units). Comparing the bases of triangles, we have $\frac{x}{8}=\frac{18}{x}$. Thus $x = 12$, and the total area is $8 + 18 + 12 + 12 = 50$.

14. Ans. 45°

假設 $BM < DK$ 和 N 是 MK 上的一點使 $BM = MN$，由 N 作一線 IN 垂直於 KM。明顯可見 $\Delta IBM \cong \Delta INM$。



Assume $BM < DK$ and N be a point on MK such that $BM = MN$. Draw a line through N perpendicular to KM. It is easy to see that $\Delta IBM \cong \Delta INM$.

$$IJ = IN - JN = IB - JD + (JD - JN)$$
$$= (AB + AI) - (AD - AJ) + (JD - JN) = AI + AJ + (JD - JN)$$
$$\therefore \quad JD - JN = IJ - (AI + AJ)$$

$\because \quad MC + CK + KM = BC + DC \Rightarrow KM = BM + DK \Rightarrow NK = DK$

$\Rightarrow \quad \Delta JKN \cong \Delta JKD \Rightarrow JD = JN \Rightarrow I = A$

因此 Therefore :

$\angle MAK = \angle MAN + \angle NAK = 45°$

15. Ans. C, D, F, G

16. Ans. $60°$

解一：設 X 和 Y 分別爲 BF 和 DH 的中點， 在 $PXSRYQ$ 正六邊形中，SQ 和 PR 是這正六邊形的對角線。

Solution 1: Consider the regular hexagon $PXSRYQ$, where X and Y are midpoints of BF and DH respectively. SQ and PR are two diagonals that pass through the center of this regular hexagon.

解二：∵ $SQ \parallel GD$ 和 $PR \parallel BG$ 及 ΔBGD是一等邊三角形

∴ SQ 與 PR間的銳角 $= GD$ 與 GB間的交角。

Solution 2: SQ is parallel to GD while PR is parallel to BG. BGD is an equilateral triangle. The angle between SQ and PR is the same as that between GD and GB.

17. Ans. 13 minutes

個案一： 4 和 6 分別行。

在任何情況下，都需要最少 4 + 1 + 6 + 1 + 2 = 14 分鐘。

個案二： 4 和 6 一起行。

在任何情況下，他們都不會是最先或最後的一組過橋，因此需要最少 2 + 1 + 6 + 2 + 2 = 13 分鐘。

Case 1: 4 and 6 travel separately.

In this case any crossing requires at least 4 + 1 + 6 + 1 + 2 = 14 minutes.

Case 2: 4 and 6 travel together.

In this case they clearly cannot be either the first or the last pair to cross, so the time required is at least 2 + 1 + 6 + 2 + 2 = 13 minutes.

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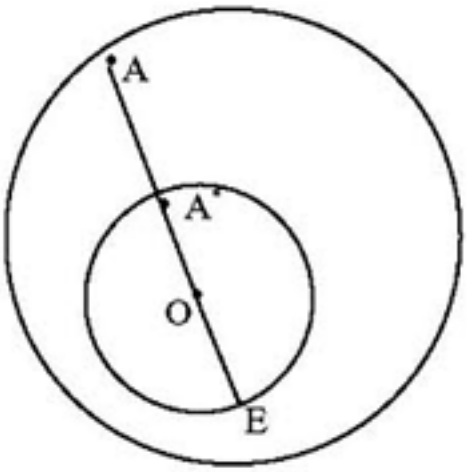
建議題解 Suggested Solutions

乙部題解 Part B

18.



明顯地，我們知道 $r \le OA$ .....(1)

$2R \ge AE$ .....(2)

$$AA^* = OA - OA^* = OA - \frac{r^2}{OA} = \frac{OA^2 - r^2}{OA} = \frac{(OA-r)(OA+r)}{OA} = (OA-r)(1+\frac{r}{OA}) \le 2(OA-r)$$ (從(1))。

由 (2), $OA + r \le 2R$。 因此 $OA - r \le 2R - 2r = 2(R-r)$。 所以。 我們推斷出 $AA^* \le 2(OA-r) \le 4(R-r)$。

Clearly, we have $r \le OA$ .....(1)

$2R \ge AE$ .....(2)

$$AA^* = OA - OA^* = OA - \frac{r^2}{OA} = \frac{OA^2 - r^2}{OA} = \frac{(OA-r)(OA+r)}{OA} = (OA-r)(1+\frac{r}{OA}) \le 2(OA-r)$$

(by (1)).

From (2), $OA + r \le 2R$. Hence $OA - r \le 2R - 2r = 2(R-r)$. Therefore, we concluded that $AA^* \le 2(OA-r) \le 4(R-r)$.

19. Ans. 135

設所需的數字爲 abc。 那麼 $100a + 10b + c = abc(a + b + c)$。 化簡以上算式，我們得到 $9(11a + b) = (a + b + c)(abc - 1)$。 嘗試 $a + b + c = 9$。 〈我們只需找出一個數字〉 因爲 $a = 1$ 而 b 和 c 不可能是 0， 所以我們得到以下 7 個情況：

Let the required number be abc. Then $100a + 10b + c = abc(a + b + c)$. Simplifying the expression, we get $9(11a + b) = (a + b + c)(abc - 1)$. Try $a + b + c = 9$. (we have to find one number only) As $a = 1$ and b , c must not be 0, we have the following 7 cases :

a	b	c	$a + b + c$	abc	$(a + b + c)abc$
1	1	7	9	7	63
1	2	6	9	12	108
1	3	5	9	15	135
1	4	4	9	16	144
1	5	3	9	15	135
1	6	2	9	12	108
1	7	1	9	7	64

所以其中一個答案是 135。

Hence, 135 is a solution.

20. Ans. 342

設 r 爲對角線上黑色階磚的數目，並且是正方形階磚圖案一邊的階磚數目。在沿著對角線的上方，那裡有 $r - 1$ 片黑色階磚，相似地，那裡亦有 $r - 1$ 片黑色階磚在下方。因此， $r + (r - 1) + (r - 1) = 58$ 而得 $r = 20$。 所以正方形階磚圖案有 20 片黑色階磚在其中一邊，並且總共有 400 片黑色及白色階磚。 而白色階磚的數目 = 400 – 58 = 342。

Let x be the number of black tiles along the diagonal, which is also the number of tiles along one side of the square pattern. There are $r - 1$ black tiles above the diagonal, and similarly another $r - 1$ below. So, $r + (r - 1) + (r - 1) = 58$ and $r = 20$. Therefore the square pattern has 20 tiles on one side and altogether 400 black and white tiles. Number of white tiles = 400 – 58 = 342.

建議題解完 End of Suggested Solutions