香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2001-2002

建議題解 Suggested Solutions

1. **Ans. 6 cm³** 將正八面體分成兩等份底爲正方形的金字塔，底面積是立方體橫切面的一半，高度是立方體的長度的一半，所以正八面體的體積是正立方體體積的六份一。

   Bisect the octahedron into two pyramids with square bases. The base area is half of the cross section of the cube. The altitude of each of the pyramids is half of the length of the cube. The volume of the octahedron is therefore 1/6 of the volume of the cube.

2. **Ans. 7.5 cm²** 以弦 EG 將四邊形分成兩個三角形。因爲 $AE = DG$，所以三角形 FEG 的面積是平行四邊形 $BEGC$ 面積的一半，相同地，另一個三角形是剩餘的平行四邊形的一半，所以 $EFGH$ 的面積是原本的平行四邊形的一半。

   The quadrilateral can be divided into two triangles by the segment EG, which is to AD and BC because $AE = DG$. Area of triangle FEG is half of the area of parallelogram $BEGC$. Similarly, the other triangle is also half of the remaining parallelogram. The area of $EFGH$ is therefore half of that of the original parallelogram.

3. **Ans. 3, 4, 5** 極限的情況爲 (1) B 是在 AC 上及 (2) B 在 CA 上。情況 (1)，$AD = \dfrac{3+8}{2} = 5.5$，情況(2)，$AD = \dfrac{3+8}{2} - 3 = 2.5$，即 $2.5 < x < 5.5$；從而得 $x = 3$ 或 4 或 5。

   The limiting cases are (1) when B is on AC and (2) B is on CA produced. For (1), $AD = \dfrac{3+8}{2} = 5.5$; For (2), $AD = \dfrac{3+8}{2} - 3 = 2.5$; $2.5 < x < 5.5$; $x = 3$ or 4 or 5.

4. **Ans. 153** 考慮 $x \geq 6$，即 $x^3 \geq 216$，所以 □ 及 Δ 必定少於 6。考慮 $x \leq 4$，即 $x^3 \leq 64$，所以 □ 或 Δ 中必有一個是 5。透過實驗嘗試 1□Δ 是 153。

$x^3 \geq 216$ for $x \geq 6$, therefore □ and Δ must be both < 6; $x^3 \leq 64$ for $x \leq 4$, therefore one of □ and Δ must be 5. By trial and error, 1 □ Δ must be 153.

5. **Ans. 6** 因爲長度及闊度是 4 cm 的倍數，設 4 cm 爲 1 單位，面積是 $= \dfrac{960}{4^2}$ 平方單位 $= 60$ 平方單位，$60 = 1 \times 60 = 2 \times 30 = 3 \times 20 = 4 \times 15 = 5 \times 12 = 6 \times 10$，所以有 6 個不同形狀。

As the length and width are multiples of 4 cm, let 4 cm be 1 unit. The area $= \dfrac{960}{4^2}$ sq. units $= 60$ sq. units. $60 = 1 \times 60 = 2 \times 30 = 3 \times 20 = 4 \times 15 = 5 \times 12 = 6 \times 10$. Therefore there are 6 different shapes.

6. **Ans. 137** 因爲只有一個雙數，而 單數 ＋ 單數 ＝ 雙數，所以，任意兩個單數的和是 66, 98 及 110。由此，所求和是 $\dfrac{66+98+110}{2} = 137$。

Since there is only one even number and we must have odd + odd = even. Hence, sums of any two odd numbers are 66, 98 and 110 . So, the required sum is $\dfrac{66+98+110}{2} = 137$.

7. **Ans. 9** 注意到 $7 \times 7 = 49$，$7 \times 7 \times 7 = 343$，$7 \times 7 \times 7 \times 7 = 2401$。 考慮各乘積的個位可得數列 7, 9, 3, 1, 7, 9, 3, 1, …。因爲 $2002 \div 4$ 的餘數爲 2，所以 7^{2002} 的個位數與 7^2 的個位數相同，即 9。

Consider $7 \times 7 = 49$, $7 \times 7 \times 7 = 343$, $7 \times 7 \times 7 \times 7 = 2401$. Looking at the unit digits we get a sequence 7, 9, 3, 1, 7, 9, 3, 1, … Thus, $2002 \div 4$ has a remainder of 2. Thus, the unit digit of 7^{2002} is the same as that of 7^2, i.e. 9.

8. **Ans. 19** 設該數爲 abc，則其餘五個數是 acb，bac，bca，cab 及 cba。因此，$4218 = 222 \times (a+b+c)$，由此，$a+b+c = 4218 \div 222 = 19$

Suppose the number is abc. Re-arranging to get abc, acb, bac, bca, cab, cba. So, $4218 = 222 \times (a+b+c)$, $a+b+c = 4218 \div 222 = 19$.

9. $\boxed{\text{Ans. 6:1}}$ 　　由於 $BD:DC = 1:2$，所以 $\triangle FBD$ 面積 : $\triangle FDC$ 面積 $= 1:2$。設 $\triangle FBD$ 面積爲 1 平方單位，則 $\triangle FBD$ 面積 及 $\triangle FBC$ 面積分別是 2 平方單位及 3 平方單位。由於 $\triangle FBC$ 及 $\triangle ABF$ 有相同的底 BF 而 $EC:EA = 1:2$，所以 $\triangle ABF$ 面積爲 6 平方單位。考慮 $\triangle FBD$ 以 FD 爲底及 $\triangle ABF$ 以 AF 爲底時有相同的高，$AF:FD = \triangle ABF$ 面積 : $\triangle FBD$ 面積 $= 6:1$。

　　　　Since $BD:DC = 1:2$, then, area $\triangle FBD$: area $\triangle FDC = 1:2$ also. Let the area of $\triangle FBD$ be 1 sq. unit, then the area of $\triangle FDC$ is 2 sq. units, and the area of $\triangle FBC$ is 3 sq. units. $\triangle FBC$ and $\triangle ABF$ share the same base BF, and $EC:EA = 1:2$. Thus, area of $\triangle ABF$ is 6 sq. units. Finally, if $\triangle ABF$ uses AF as base, and $\triangle FBD$ uses FD as base, then they share the same height. Area$\triangle ABF$: area$\triangle FBD = 6:1$; hence, $AF:FD = 6:1$.

10. $\boxed{\text{Ans. \$650}}$ 　　20 張小童票於折扣下相當於 18 張小童票價。因此，18 小童票 + 3 成人票 共值 9150，而 7 小童票 + 2 成人票 共值 4100。解此聯立方程可知每張成人票價爲 \$650。

　　　　20 children at discount rate is equivalent to 18 children at regular rate. Thus, 18 children + 3 adults cost 9150, and 7 children + 2 adults cost 4100. Solving the simultaneous equations to get 650 as the cost for an adult.

11. $\boxed{\text{Ans. 7 sq. units}}$ 　　考慮 $\triangle ABC$ 及 $\triangle FCE$，它們擁有相同的底($BC = CE$)。因爲 $2CA = CF$，所以 $\triangle FCE$ 的高是 $\triangle ABC$ 的高的兩倍。由此，$\triangle FCE$ 面積 $= 2 \times \triangle ABC$ 面積。同理，可得 $\triangle ADF$ 及 $\triangle BDE$ 的面積。

　　　　Consider only $\triangle ABC$ and $\triangle FCE$. Their bases have the same length ($BC = CE$). Since $2CA = CF$, the height of $\triangle FCE$ is twice that of $\triangle ABC$. Thus, area $\triangle FCE = 2 \times$ area $\triangle ABC$. Similar for $\triangle ADF$ and $\triangle BDE$.

12. $\boxed{\text{Ans. 6600m}}$ 　X 和 Y 的步速每分鐘相差 30m。所以當 Z 遇到 X 的時候，Y 是落後 X 600m。再兩分鐘後，Y 遇到 Z。即 Y 和 Z 在這兩分鐘共步行了 600m。由於 Y 的步速爲每分鐘 130m，故 Z 每分鐘的步速爲 300-130，即每分鐘 170m。由於 X 和 Z 能 20 分鐘「環繞」水塘一周，而他們的步速「共」爲每分鐘 160+170=330m，故此水塘的周界爲 330×20 = 6600 m。

The difference of speed of X and Y is 30m/min. Thus, Y was 600m behind X when Z met X. After 2 more minutes Y met Z. They walked for 600m in 2 minutes. Since the speed of Y is 130m/min, it means that the speed of Z is $300 - 130 = 170$m/min. X and Z combined have a "speed" of $160 + 170 = 330$ m/min and they completed the perimeter in 20 minutes. Thus the perimeter is $330 \times 20 = 6600$.

13. Ans. $\boxed{\dfrac{1}{24}}$ $\dfrac{x^2}{x^4 + x^2 + 1} = \dfrac{1}{\left(x + \dfrac{1}{x}\right)^2 - 1} = \dfrac{1}{5^2 - 1}$.

14. Ans. $\boxed{\dfrac{45}{4}}$ 基於對稱，EF 會穿過 AC 的中點 O，另 EF 垂直 AC。明顯地，

$\triangle AOE \sim \triangle ABC$。$\dfrac{EO}{AO} = \dfrac{9}{12} = \dfrac{3}{4}$，$AO = \dfrac{15}{2}$，$\therefore EO = \dfrac{45}{8}$。故 $EF = \dfrac{45}{4}$。

By symmetry, EF passes through the midpoint O of AC and is perpendicular to AC. Clearly, $\triangle$ AOE $\sim$ $\triangle$ ABC. $\dfrac{EO}{AO} = \dfrac{9}{12} = \dfrac{3}{4}$, $AO = \dfrac{15}{2}$, $\therefore EO = \dfrac{45}{8}$. Thus $EF = \dfrac{45}{4}$.
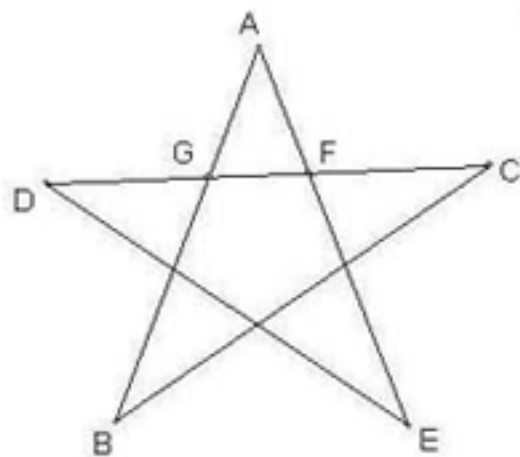
15. Ans. $\boxed{33334}$ 11111 和 3 皆為 1111122222 的因數。故此，1111122222 = 33333×33334。

1111122222 has factors 11111 and 3. Hence, 1111122222 = 33333×33334.

16. Ans. $\boxed{24°}$ $\angle ADC = \angle C = \angle B + \angle BAD$。由於 $\angle B = \angle BAD$，所以 $\angle C = 2\angle B$。再者，$\angle A + \angle B + \angle C = 63° + 3\angle B = 180°$。所以，$\angle B = 39°$，$\angle C = 78°$。而 $\angle DAC = 180° - 2\angle C = 24°$。

$\angle ADC = \angle C = \angle B + \angle BAD$. Since $\angle B = \angle BAD$, so $\angle C = 2\angle B$. Moreover, $\angle A + \angle B + \angle C = 63° + 3\angle B = 180°$. So, $\angle B = 39°$, $\angle C = 78°$. $\angle DAC = 180° - 2\angle C = 24°$.

17. 　Ans. 44°　 考慮$\triangle DEF$，$\angle DFE = 180° - 2 \times 34° = 112°$，所以$\angle AFG = 68°$。用相同方法，我們會得到$\angle AGF = 68$。故此，$\angle A = 44°$。

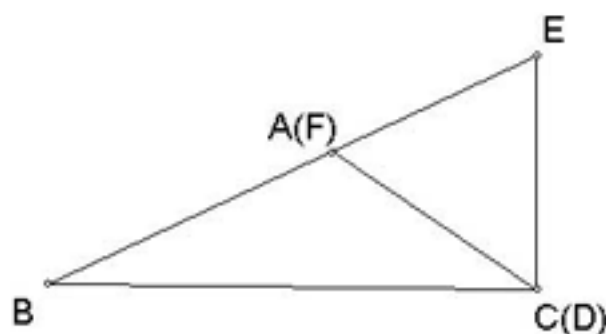　　　Consider triangle DEF, $\angle DFE = 180° - 2 \times 34° = 112°$, so $\angle AFG = 68°$. Similarly, $\angle AGF = 68°$. So, $\angle A = 44°$

Part B （乙部）

把完整的題解和答案寫在答題紙所提供的位置

Answer the following questions completely on the spaces provided in the answer sheet.

18. 　Ans. 9.6 cm²　 在 $\triangle DEF$, 由 $DE = DF$ 可知 $\angle E = \angle F$，$\angle D = 180° - 2 \times 63° = 54°$. 因為 $AC = DF$ 及 $\angle ACB + \angle EDF = 90°$，而 $\angle BAC + \angle DFE = 180°$，$\triangle ABC$ 和 $\triangle DEF$ 能夠組成一個直角三角形，$BC = BD = 8$ 及 $CE = DE = 4$。所以新的直角三角形面積是 16 cm²，並且三角形 ABC 的面積是 $16 - 6.4 = 9.6$ cm².

　　　In $\triangle DEF$, $DE = DF$ implies $\angle E = \angle F$, $\angle D = 180° - 2 \times 63° = 54°$. Since $AC = DF$ and $\angle ACB + \angle EDF = 90°$, and $\angle BAC + \angle DFE = 180°$, $\triangle ABC$ and $\triangle DEF$ can be combined to form a right-angled triangle. $BC = BD = 8$ and $CE = DE = 4$. Thus, the area of the new right-angled triangle is 16 cm², and the area of triangle ABC is $16 - 6.4 = 9.6$ cm².

19. 　Ans. (b) 11 cm　 設邊長為 1 cm 的正三角形面積為 x，則以 2 cm、3 cm、4 cm 及 5 cm 為邊長的正三角形面積分別是 4x、9x、16x 及 25x。由此，總面積是 $(4 \times 3 + 9 \times 4 + 16 \times 3 + 25 \times 1) \times x = 121x$。所以大正三角形的邊長為 11 cm。
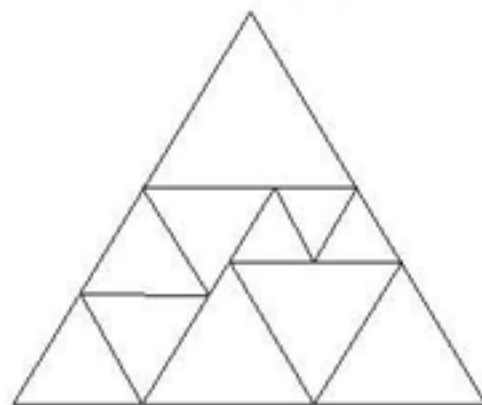
　　　Let the area of the equilateral triangle with side 1 cm be x, then the areas of equilateral triangles with side 2 cm, 3 cm, 4 cm and 5 cm are 4x, 9x, 16x, and 25x respectively. The total area is $(4 \times 3 + 9 \times 4 + 16 \times 3 + 25 \times 1) \times x = 121x$. So, the new equilateral triangle should have side 11 cm.

20. (a) 　Ans. 28　 不通過 B 點：

　　　　　Not passing through B:

若先通過 C，再通過 D，

If D comes after C, we have

A → C → D → A,   A→ C → D → E → A,

A → C → D → E → F → A.

若先通過 C，再通過 F，得出通過路線的總數仍是一樣。所以共有
(3＋3＋1)×4 = 28 不同的通過路線(因為還有 A → C→ A)。

If F comes after C, the number is the same.   So, there are (3＋3＋1)×4 = 28
different paths. (The last one is the case A → C→ A).

(b) Ans. 76   通過 B 點：

Passing through B:

先經過 C，再直接到 B 的路線：

Going from C directly to B, the paths are

A → C → B → (D or E or F) → A,

A → C → B → (D → E or E → D or E → F or F → E) → A,

A → C → B → (D → E → F or F → E → D) → A.

共有 9 種不同的通過路線。

There are 9 different choices.

先經過 C，再經過其他頂點才到達 B 的通過路線：

Also, going from C through other points to B, the paths are

A → C → D → B → (E or F) → A,

A → C → F → B → (D or E) → A,

A → C → D → B → (E → F or F → E) → A,

A → C → F → B →(E → D or D → E) → A,

A → C → D → E → B → F→ A,

A → C -> F → E → B → D → A.

共有 10 種不同的通過路線。故此共有 4×(9＋10) = 76 不同的通過路線。

There are 10 more choices.   The total is thus 4×(9＋10) = 76.