香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2000 – 2001

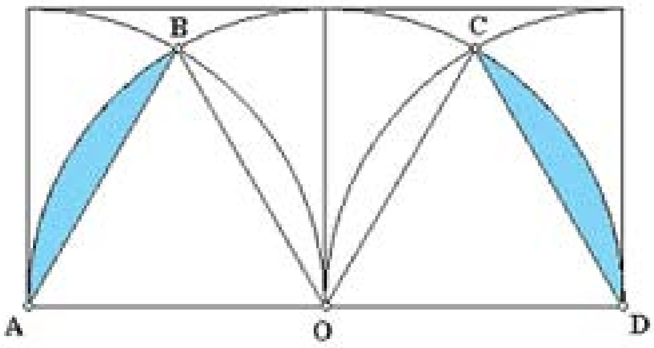
建議題解 *Suggested Solutions*

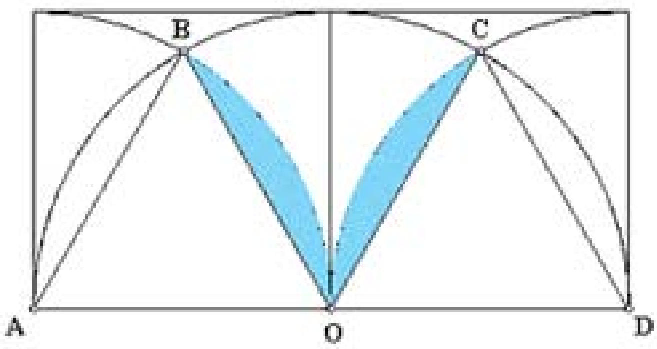
1. *題解*：四個正方體共有二十四面，有六面（即三對）隱藏了，所以總表面面積是 $\boxed{18}$ cm^2。

   Solution: Total number of squares on the surface of the 4 cubes is 24, 6 of them (3 pairs) are hidden. The total surface area is $\boxed{18}$ cm^2.

2. *題解*：弦 AB 及弦 CD 上的弓形與弦 BO 及弦 CO 上的弓形面積相等，所以所需面積是扇形 OBC 的面積，$\frac{60^\circ}{360^\circ}\cdot\pi\cdot(1^2)cm^2=\frac{\pi}{6}cm^2$，而 $x=\boxed{1/6}$。

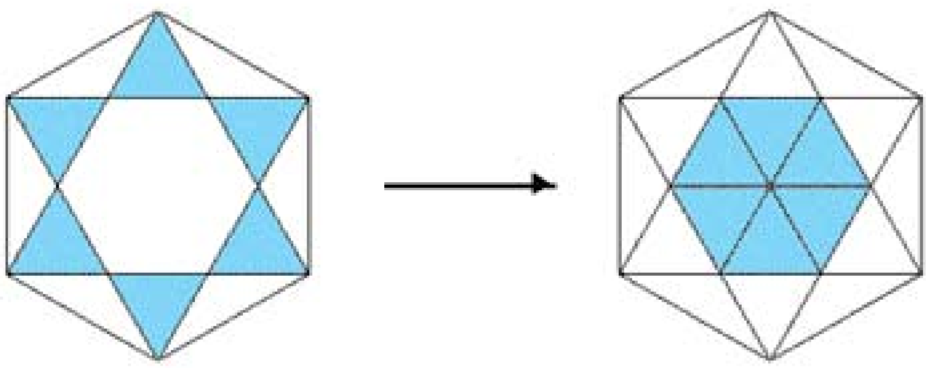
   Solution: The segments on arcs *AB* and *CD* are equal to those on *BO* and *CO*. Therefore the required area is the same as the area of sector *OBC*, which is $\frac{60^\circ}{360^\circ}\cdot\pi\cdot(1^2)cm^2=\frac{\pi}{6}cm^2$, and $x=\boxed{1/6}$.

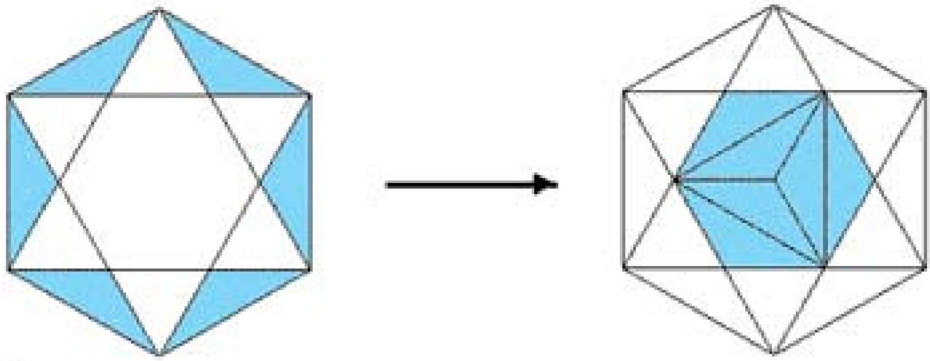




3. *題解*：由下圖可見大六邊形的面積是小六邊形的三倍，即 $\boxed{15}$ cm^2。

   Solution: The area of the larger hexagon is 3 times that of the smaller hexagon, and is therefore $\boxed{15}$ cm^2. The following diagram is self-explanatory.





4. *題解*：如下式。

$$\sqrt[64]{(2+1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=\sqrt[64]{(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=$$

$$\sqrt[64]{(2^2-1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=\sqrt[64]{(2^4-1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}$$

$$=\sqrt[64]{(2^{128}-1)+1}=\boxed{4}$$ 。

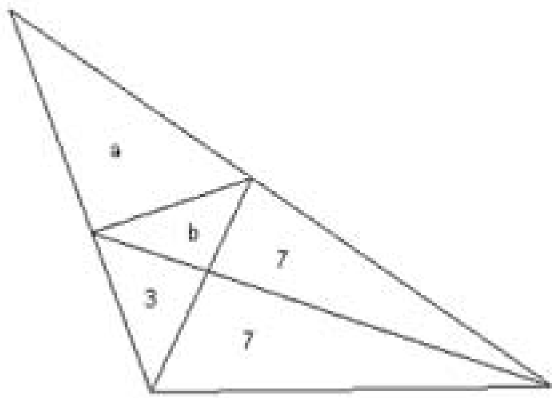
Solution: Simplify as follows

$$\sqrt[64]{(2+1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=\sqrt[64]{(2-1)(2+1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=$$

$$\sqrt[64]{(2^2-1)(2^2+1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}=\sqrt[64]{(2^4-1)(2^4+1)(2^8+1)\Lambda\ (2^{64}+1)+1}$$

$$=\sqrt[64]{(2^{128}-1)+1}=\boxed{4}.$$

5. *題解*：$\frac{a+b+3}{7+7}=\frac{a}{b+7}$ ----- (1)



$\frac{3}{7}=\frac{b}{7}$ ----- (2)

由 (2)， $b=3$

所以，由(1)， $10(a+6)=a(14)$

$\therefore\quad a=15$

所以， $t=a+b=15+3=\boxed{18}$ 。

Solution: $\frac{a+b+3}{7+7}=\frac{a}{b+7}$ ----- (1)

$\frac{3}{7}=\frac{b}{7}$ ----- (2)

From (2), $b=3$

and hence, from (1), $10(a+6)=a(14)$

$\therefore\quad a=15$

Therefore, $t=a+b=15+3=\boxed{18}$.

6. *題解*：設象限 OPQ 的半徑爲 $2r$，

象限的面積 =（OP 及 OQ 上的兩個半圓的面積）+（$b-a$）

得出 $\frac{1}{4}\pi(2r)^2 = \frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2 + b - a$

$\therefore \quad a = b$

所以 $\frac{a}{b} = \boxed{1}$。

Solution: Suppose the radius of the quadrant OPQ is $2r$.

The area of the quadrant = (area of semi-circles on OP and OQ) + ($b-a$)

This gives $\frac{1}{4}\pi(2r)^2 = \frac{1}{2}\pi r^2 + \frac{1}{2}\pi r^2 + b - a$

$\therefore \quad a = b$.

That is $\frac{a}{b} = \boxed{1}$

7. *題解*：$\boxed{\sqrt{12^2 - 5^2} = \sqrt{119}}$。

Solution: $\boxed{\sqrt{12^2 - 5^2} = \sqrt{119}}$.

8. *題解*：$E(23046) = \boxed{3}$，$E(102) = \boxed{1}$。

$E(1) + E(2) + E(3) + \Lambda + E(98) + E(99)$

$= (1+3+5+7+9)\times 10 + (1+3+5+7+9)\times 10 = \boxed{500}$。

Solution: $E(23046) = \boxed{3}$, $E(102) = \boxed{1}$.

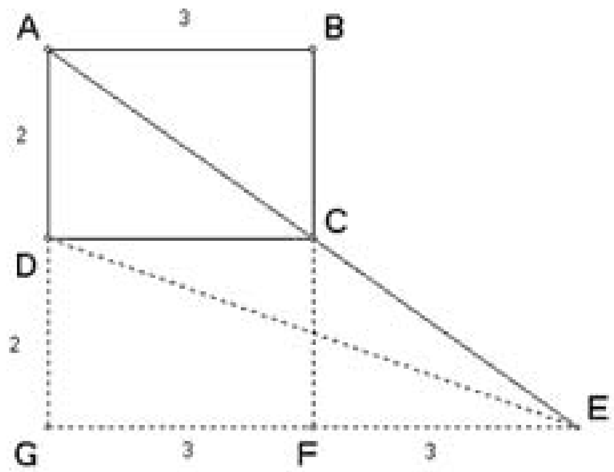
$E(1) + E(2) + E(3) + \Lambda + E(98) + E(99)$

$= (1+3+5+7+9)\times 10 + (1+3+5+7+9)\times 10 = \boxed{500}$.

9. *題解*：如圖示，易見 $\boxed{DE = \sqrt{2^2 + 6^2} = \sqrt{40}}$。

Solution: It is easy to see from the figure that

$\boxed{DE = \sqrt{2^2 + 6^2} = \sqrt{40}}$.



10. *題解*：48 個。因爲 0 不可用作百位數，只有 4 個數字可用，然後有 4 個可用作十位數，三個作個位數。$4 \times 4 \times 3 = \boxed{48}$。

Solution: 48 numbers. Only four of the digits are possible for the hundreds digit because 0 cannot be used in that position. Then four digits are available for the tens digit and three for the units digit. $4 \times 4 \times 3 = \boxed{48}$.

11. *題解*：$2 \& 3 = 2^3 + 3^2 = 17$；
$17 \& 2 = 17^2 + 2^{17} = 289 + 131072 = \boxed{131361}$。
Solution: $2 \& 3 = 2^3 + 3^2 = 17$
$17 \& 2 = 17^2 + 2^{17} = 289 + 131072 = \boxed{131361}$.

12. *題解*：個位數是以 3、9、7、1、3、9、7、1、……的方式循環，所以3^{33}的個位數是$\boxed{3}$。
Solution: The units digit follows the cycle 3, 9, 7, 1, 3, 9, 7, 1, …; hence the units digit of 3^{33} is $\boxed{3}$.

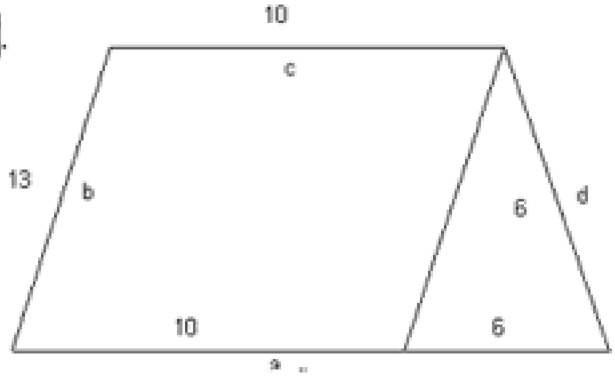
13. *題解*：$2^{800} = 16^{200}$，$3^{600} = 27^{200}$，$5^{400} = 25^{200}$，所以$\boxed{6^{200} < 2^{800} < 5^{400} < 3^{600}}$。
Solution: $2^{800} = 16^{200}$, $3^{600} = 27^{200}$, $5^{400} = 25^{200}$；hence, $\boxed{6^{200} < 2^{800} < 5^{400} < 3^{600}}$.

14. *題解*：1 和 200 之間共有 $\lfloor \sqrt{200} \rfloor = 14$ 個完全平方和 $\lfloor \sqrt[3]{200} \rfloor = 5$ 個完全立方，中間有 $\lfloor \sqrt[6]{200} \rfloor = 2$ 個（1 和 64）被數了兩次，所以總共有 $14 + 5 - 2 = 17$ 個在 1 和 200 間的整數不在數列內。我們需要在 2、3、5、……、200 這 173 個非平方及非立方後加上 17 個整數，由於 216 不能被使用，最後一個數是$\boxed{218}$。

Solution: Between 1 and 200, there are $\lfloor \sqrt{200} \rfloor = 14$ perfect squares and $\lfloor \sqrt[3]{200} \rfloor = 5$ perfect cubes. Among these integers there are $\lfloor \sqrt[6]{200} \rfloor = 2$ of them (1 and 64) that are counted twice. Thus there are $14 + 5 - 2 = 17$ integers between 1 and 200 that are not in the sequence. To get the 200^{th} number, we must append 17 integers to the list 2, 3, 5, … , 200 of 173 non-squares and non-cubes. Since we cannot use 216, the last number will be $\boxed{218}$.

15. *題解*：若能構作符合這些條件的四邊形， 則圖中三角形的三條邊分別是 6、6，及 13 單位長，但兩邊的長度的和小於第三邊的長度，所以符合這些條件的四邊形並不存在。即$\boxed{0}$個。

Solution: If such a quadrilateral can be constructed, then the lengths of the sides of the triangle in the figure are 6, 6, and 13 respectively, but the sum of lengths of two sides is smaller than that of the third side. Thus no such quadrilateral can be constructed, i.e. $\boxed{0}$.



16. *題解*：使用對數得 $x\log 3 = y\log 4 = z\log 12$ ，再代入求得 $\boxed{\frac{z}{x}+\frac{z}{y}=1}$。

Solution: Apply logarithms on the equations to get $x\log 3 = y\log 4 = z\log 12$. Then substitute to get $\boxed{\frac{z}{x}+\frac{z}{y}=1}$.

17. *題解*：設 $x = \underset{2001位}{11\Lambda 1}$，則

$$\sqrt{\underset{4002位}{11\Lambda 1} - \underset{2001位}{22\Lambda 2}} = \sqrt{\underset{2002位}{100\Lambda 0}x + x - 2x} = \sqrt{\underset{2001位}{99\Lambda 9}x} = \sqrt{9\times x^2} = 3x。$$

答案是 $\boxed{\underset{2001位}{33\Lambda 3}}$。

Solution: Let $x = \underset{2001digits}{11\Lambda 1}$, then

$$\sqrt{\underset{4002digits}{11\Lambda 1} - \underset{2001digits}{22\Lambda 2}} = \sqrt{\underset{2002digits}{100\Lambda 0}x + x - 2x} = \sqrt{\underset{2001digits}{99\Lambda 9}x} = \sqrt{9\times x^2} = 3x.$$

The answer is $\boxed{\underset{2001digits}{33\Lambda 3}}$.

Part B（乙部）

1. *題解*：由於 $\frac{n^3+n-4}{n-4}=n^2+4n+17+\frac{64}{n-4}$，

若 R.H.S. 爲整數，則 64 必爲 $n-4$ 除盡；若 n 是正數，$n-4>-3$。

64 的因數中大於 -3 的有 -2、-1、1、2、4、8、16、32、64，

所以 $n=\boxed{2、3、5、6、8、12、20、36、68}$。

Solution: Since $\frac{n^3+n-4}{n-4}=n^2+4n+17+\frac{64}{n-4}$.

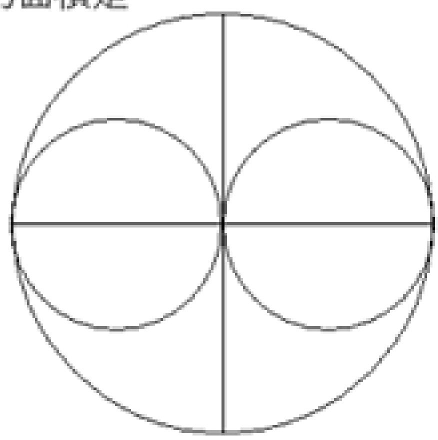
R.H.S. is an integer only if 64 is divisible by $n-4$.

If n is positive, $n-4>-3$.

Factors of 64 which are greater than -3 are -2, -1, 1, 2, 4, 8, 16, 32, 64.

Therefore $n=\boxed{2, 3, 5, 6, 8, 12, 20, 36, 68}$.

2. *題解*：構作半徑及圓形如圖。大圓的面積是 $\pi(1)^2$，而兩個小圓的面積是 $\pi\left(\frac{1}{2}\right)^2+\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{2}$。所以，在大圓內而位於小圓外的面積是 $\frac{\pi}{2}$，由於對稱性，所有八個區域的面積都是 $\frac{\pi}{8}$。



Solution: Construct diameters and circles as in the figure.

The area of the large circle is $\pi(1)^2$, and the combined area of the two small circles is $\pi\left(\frac{1}{2}\right)^2+\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{2}$. Thus, the area of that portion of the large circle that lies outside the small circles is also $\frac{\pi}{2}$, and by symmetry, all eight regions have the same area of $\frac{\pi}{8}$.

3. *題解*：由於 3^c+1 必定是雙數，所以它的其中一個因數 (2^a+1) 或 (4^b+1) 必定是雙數，由此可得 a 或 b 必定等於 0；如果 $a=0$ 而 $b=1$，則 $c=2$，而總和是 3；但是，如果 $b=0$，a 和 c 的值（如果它們存在）會是非常之大；所以 $a+b+c=\boxed{3}$。

Solution: Since 3^c+1 must be even, at least one of the factors (2^a+1) or (4^b+1) must be even. For this to be true, either a or b must be 0. If $a=0$ and $b=1$, then $c=2$ and their sum is 3. However, if $b=0$, the values for a and c (if they exist) would be much larger. Thus $a+b+c=\boxed{3}$.