香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

1998-1999

建議題解　Suggested Solution

Part A

1. Let x=8.5, y=3.5. The expression becomes

$$\sqrt[3]{(x^2-y^2)(x^2-2xy+y^2)(x^2+2xy+y^2)}$$
$$=\sqrt[3]{(x^2-y^2)(x-y)^2(x+y)^2}$$
$$=\sqrt[3]{(x-y)^3(x+y)^3}$$
$$=(x-y)(x+y)$$
$$=60$$

2. $$\frac{(2\#3)\#5}{2\#(3\#5)}=\frac{(2\times3-1)\#5}{2\#(3\times5-1)}=\frac{5\#5}{2\#14}=\frac{5\times5-1}{2\times14-1}=\frac{8}{9}$$

3.

$$x^2=\frac{1}{4}\left(\sqrt[3]{7^2}-2+\frac{1}{\sqrt[3]{7^2}}\right)$$
$$1+x^2=\frac{1}{4}\left(\sqrt[3]{7^2}+2+\frac{1}{\sqrt[3]{7^2}}\right)$$
$$\sqrt{1+x^2}=\frac{1}{2}\left(\sqrt[3]{7}+\frac{1}{\sqrt[3]{7}}\right)$$
$$x+\sqrt{1+x^2}=\frac{1}{2}\left(\sqrt[3]{7}-\frac{1}{\sqrt[3]{7}}\right)+\frac{1}{2}\left(\sqrt[3]{7}+\frac{1}{\sqrt[3]{7}}\right)=\sqrt[3]{7}$$
$$\left(x+\sqrt{1+x^2}\right)^3=7$$

4. Let x=1998. The expression becomes

$$(x-2)(10000x+x)-x(10000(x-2)+x-3)$$
$$=(x-2)10000x+(x-2)x-(x-2)10000x-x(x-3)$$
$$=x^2-2x-x^2+3x$$
$$=x=1998$$

5. Quadrilaterals ABCM and BCDK are congruent. Since BCML is common to both quadrilaterals, therefore, the remaining regions ABL and MLKD should have the same area. That means area of MLKD = 16 cm^2.

6. Let O be the origin, A = (5,0), B = (0,5) and C = (-4,-3).
Area of the quarter OAB = $\frac{1}{4}\pi \cdot 5^2 = \frac{25\pi}{4}$
Area of $\Delta OAC = \frac{1}{2}(5)(3) = \frac{15}{2}$, (with base = 5 and height = 3),
and area of $\Delta OBC = \frac{1}{2}(5)(4) = 10$, (with base = 5 and height = 4).
Therefore, area of shaded region = $\frac{25\pi}{4} + \frac{15}{2} + 10 = \frac{25\pi + 70}{4}$.

7. Since the longest side of a triangle must be less than the sum of the other two sides, it follows that $4 < n < 26$. For the triangle to be obtuse, either $11^2 + 15^2 < n^2$ or $11^2 + n^2 < 15^2$. There are 13 suitable values of n : 5, 6, 7, 8, 9, 10, 19, 20, 21, 22, 23, 24 and 25.

8. The number 1998 has prime factorization $2 \times 3^3 \times 37$. It has eight factor pairs, among which the smallest difference is 54 – 37 = 17.

9. Since the degree measure of an interior angle of a regular *n*-sided polygon is $(n-2) \times \frac{180}{n} = 180 - \frac{360}{n}$, it follows that *n* must be a divisor of 360. Since $360 = 2^3 \times 3^2 \times 5$, there are $4 \times 3 \times 2 = 24$ divisors of 360. Since *n* is at least 3, we exclude the divisors 1 and 2, so there are 22 possible values of *n*.

10. The result is the same as E(00) + E(01) + E(02) + ... + E(99), which is the same as E(000102......99). There are 200 digits, and each digit occurs 20 times, so the sum of the odd digits is $20 \times (1 + 3 + 5 + 7 + 9) = 500$.

11. Let r be the radius of each sphere. Note that the centres of the eight outer spheres form a cube of side 1 - 2r in which the main diagonal is 4r units. Since the length of the diagonal of a cube is $\sqrt{3}$ times the length of its side, $\sqrt{3}$ (1 - 2r) = 4r.
Solving this equation gives r = $\frac{2\sqrt{3}-3}{2}$ or $\frac{\sqrt{3}}{2\sqrt{3}+4}$.

12. Since m and n must both be positive, it follows that n > 2 and m > 4. Because 4/m + 2/n = 1is equivalent to (m – 4)(n – 2) = 8, we need only find all the ways of writing 8 as a product of positive integers. The four ways, 1×8, 2×4, 4×2 and 8×1, correspond to the four soultions (m, n) = (5, 10) , (6, 6), (8, 4) and (12, 3).

13. Since the area of the rectangle is $9 \times 16 = 144$. Therefore the length of the side of the square is $\sqrt{144} = 12$.
$\therefore$ perimeter = $12 \times 4 = 48$.

14. area of OAQD = area of ΔOQD + area of sector OAQ

$$= 1/2OD\times DQ + 1/2OA^2\times\angle AOQ$$

$$= \frac{1}{2}(1)(\sqrt{3}) + \frac{1}{2}(2^2)\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

area of shaded region = area of quarter OAB – area of OAQD – area of OCRB + area of square OCPD

$$= \pi - 2\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) + 1 = \frac{\pi}{3} - \sqrt{3} + 1$$

15. area of quadrilateral BCED = 1/2(DF×BE + FC×BE) = 1/2(DF + FC)BE

= 1/2DC×BE = 21

area of BCED = 3/4 of area of ΔABC

$\therefore$ area of ABC = 4/3× 21 = 28

Part B

16. Since there are 8 teams, there are seven rounds of 4 matches and thus a total of $7\times 4\times 2 = 56$ points available.
Consider a team with 10 points. It is possible to have 5 teams on 10 points and 3 teams on 2 points when each of the top 5 draws with each other, each of the bottom 3 draws with each other and each of the top 5 wins against each of the bottom 3. So 10 points does not guarantee a place in top 4. Consider a team with 11 points. If this team was 5th then the number of points gained by the top 5 teams is at least 55. This is impossible as the number of points shared by the bottom 3 is then 1, as these three teams must have at least $3\times 2 = 6$ points between them for the games played between themselves. Hence 11 points is sufficient to ensure a place in the top 4.

17. After step one, twenty $3\times 3\times 3$ cubes remain, eight of which are corner cubes and twelve of which are edge cubes. At this stage each of $3\times 3\times 3$ corner cubes contributes 27 units of area and each $3\times 3\times 3$ edge cube contributes 36 units of area. The second stage of the tunneling process takes away 3 units of area from each of the eight $3\times 3\times 3$ corner cubes (1 for each exposed surface) but adds 24 units to the area (4 units for each of the six $1\times 1\times 1$ centre facial cubes removed). The twelve $3\times 3\times 3$ edge cubes each loses 4 units but gain 24 units. Therefore, the total surface area of the figure is $8\times(27 - 3 + 24) + 12\times(36 - 4 + 24) = 1056$ square units.