香港青少年數學精英選拔賽

The Hong Kong Mathematical High Achievers Selection Contest

2009 – 2010

時限：兩小時

Time allowed: 2 hours

除特別指明外，數值答案應用真確值表示。

Unless otherwise specified, numerical answers should be exact.

甲部Part A

把答案填在答題紙所提供的位置。

Write the answers on the spaces provided in the answer sheet.

1. 若 m 滿足 $(2010-x)^2+(x-2009)^2=2011$，$K=(2010-m)(m-2009)$，求 K 的值。

   If m satisfies $(2010-x)^2+(x-2009)^2=2011$, $K=(2010-m)(m-2009)$，find the value of K.

2. 已知$(a-20)^2+\sqrt{b+10}+\dfrac{c^2}{2010}=0$。若 $M=20ab-10bc$ ，求M 的值。

   It is given that $(a-20)^2+\sqrt{b+10}+\dfrac{c^2}{2010}=0$. If $M=20ab-10bc$, find the value of M.

3. 化簡 $a+b(1+a)+c(1+a)(1+b)+d(1+a)(1+b)(1+c)-(1+a)(1+b)(1+c)(1+d)$ 。

   Simplify $a+b(1+a)+c(1+a)(1+b)+d(1+a)(1+b)(1+c)-(1+a)(1+b)(1+c)(1+d)$.

4 $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{1}$, $\frac{1}{3}$, $\frac{2}{2}$, $\frac{3}{1}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{1}$, $\frac{1}{5}$, $\frac{2}{4}$, $\frac{3}{3}$, $\frac{4}{2}$, $\frac{5}{1}$, $\frac{1}{6}$, ……

   設以上數列的第 n 項為 T(n)，即 T(1) = $\frac{1}{1}$，T(2) = $\frac{1}{2}$，T(3) = $\frac{2}{1}$等。

   若 T(n) = $\frac{10}{2010}$ ，求 n 的值。

   $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{1}$, $\frac{1}{3}$, $\frac{2}{2}$, $\frac{3}{1}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{1}$, $\frac{1}{5}$, $\frac{2}{4}$, $\frac{3}{3}$, $\frac{4}{2}$, $\frac{5}{1}$, $\frac{1}{6}$, ……

   Suppose that n^{th} term of the above sequence is T(n), i.e. T(1) = $\frac{1}{1}$，T(2) = $\frac{1}{2}$，T(3) = $\frac{2}{1}$,etc.

   If T(n) = $\frac{10}{2010}$, find the value of n.

5 計算 $\dfrac{1+2^2+2^4+2^6+\Lambda\ +2^{100}}{1+2^3+2^6+2^9+\Lambda\ +2^{99}}$ 。

Calculate $\dfrac{1+2^2+2^4+2^6+\Lambda\ +2^{100}}{1+2^3+2^6+2^9+\Lambda\ +2^{99}}$.

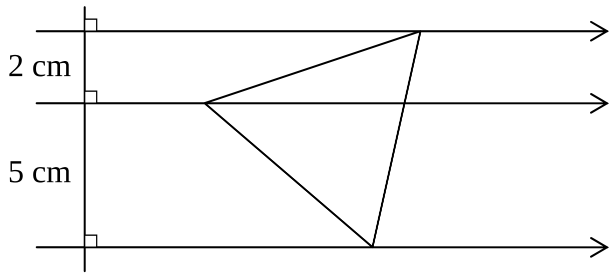
6 若 $x^2-2010x+1=0$ ，$A=x^2+\dfrac{1}{x^2}$ 。求 A 的值。

If $x^2-2010x+1=0$, $A=$ $x^2+\dfrac{1}{x^2}$. Find the value of A.

7 一個凸 2010 邊形至多能有多少個銳角?

What is the greatest possible number of acute angles a convex 2010-sided polygon can have?

8 圖中三條平行線的距離分別爲 2 cm 及 5 cm。在三條線上各取一點形成一個邊長為 x cm 的等邊三角形。求 x 的值。



The figure shows three parallel lines which are 2 cm and 5 cm apart respectively. A point is chosen on each of the lines to form an equilateral triangle with each side of x cm long. Find the value of x.

9 一個邊長爲 n (n 爲整數) 的大正方體是由若干個邊長爲 1 的正方體組成，若大正方體的 4 條對角線共穿過 33 個正方體，求 n 的值。

A big cube with each side of n units long (n is an integer) is formed from a certain number of unit cubes. If the four diagonals of the big cube pass through 33 unit cubes, find the value of n.

10 若 2010 的所有不同正因數的積爲 A，求 A 的值。(答案可用指數形式表示)

If A is the product of all the distinct positive factors of 2010. Find the value of A. (Answer in index form is accepted)

11 已知在一數列 x_1、x_2、x_3、…、x_n、… 中，$x_1 = 0$，$x_2 = 2x_1 + 1$，$x_3 = 2x_2 + 1$，…，$x_{n+1} = 2x_n + 1$，求 $x_{2010} - x_{2009}$ 的個位數字？

For the given sequence $x_1, x_2, x_3, \ldots, x_n, \ldots$, it is known that $x_1 = 0$, $x_2 = 2x_1 + 1$, $x_3 = 2x_2 + 1, \ldots, x_{n+1} = 2x_n + 1$, find the unit digit of $x_{2010} - x_{2009}$.

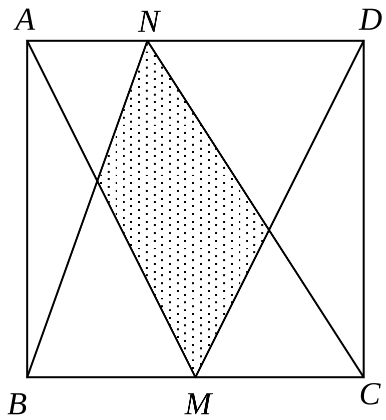
12 邊長爲整數的三條邊可以組成多少個周界爲 27 的三角形？

How many triangles of perimeter 27 units can be formed with sides of integral units？

13 若 x 及 y 爲正整數，且 $\frac{1}{x} + \frac{1}{y} = \frac{1}{2010}$，求 x 的最大値。

If x and y are positive integers and $\frac{1}{x} + \frac{1}{y} = \frac{1}{2010}$, find the greatest value of x.

14 *ABCD* 是一個正方形，*M* 是 *BC* 的中點，*AN* : *ND* = 1 : 2。
求陰影部份面積與 *ABCD* 面積的比。

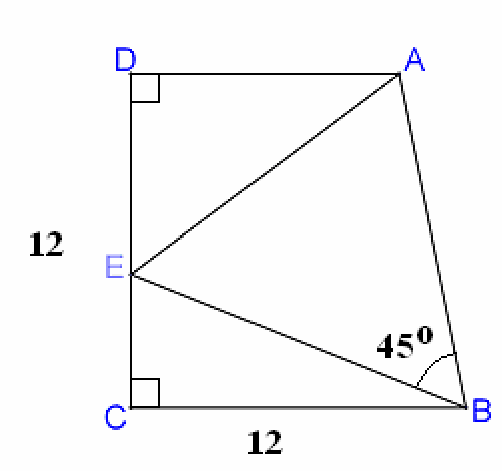


The figure shows a square *ABCD*, *M* is the midpoint of *BC* and *AN* : *ND* = 1 : 2
Find the ratio of the area of the shaded region to the area of *ABCD*.

15 若 $(x^2 + x + 1)^{1005} = a_{2010}x^{2010} + a_{2009}x^{2009} + a_{2008}x^{2008} + \Lambda + a_2x^2 + a_1x + a_0$，求 $a_{2010} + a_{2008} + a_{2006} + \Lambda + a_4 + a_2 + a_0$ 的值。 (答案可用指數形式表示)

If $(x^2 + x + 1)^{1005} = a_{2010}x^{2010} + a_{2009}x^{2009} + a_{2008}x^{2008} + \Lambda + a_2x^2 + a_1x + a_0$, find the value of $a_{2010} + a_{2008} + a_{2006} + \Lambda + a_4 + a_2 + a_0$. (Answer in index form is accepted)

16 在直角梯形 ABCD 中，BC = CD =12，E 是 CD 上的一點及∠ABE = 45°。若 AE = 10，求 CE 的長度。



In the right-angled trapezium ABCD, BC = CD =12, E is a point on CD and ∠ABE = 45°. If AE = 10，find the length of CE.

17 三個連續偶數的積爲 201****0，其中每個 * 代表某個正整數。求該三個數。

The product of three consecutive even integers is 201****0, where each * stands for some positive integer. Find these three integers.

18 在三角形 ABC 中，D 和 E 分別是 BC 和 AC 的中點，AD 和 BE 相交於 M，使三角形分成四個面積分別爲 a, b, c 和 d 的部分 ABM, BDM, DCEM 及 AEM。將 a, b, c, d 由小至大排列。

ABC is a triangle with D and E as mid-points of BC and AC respectively. AD and BE intersect at M. The triangle is divided into four regions ABM, BDM, DCEM and AEM with area a, b, c and d respectively. Arrange a, b, c, d in ascending order.

乙部 Part B

把完整的題解和答案寫在答題紙所提供的位置。

Answer the following questions with full solutions on the spaces provided in the answer sheet.

19 已知一個梯形的四條邊的邊長為 1、2、3、4，求該梯形至大可能的面積。
It is known that the lengths of a trapezium are 1, 2, 3 and 4. Find the greatest possible area of the trapezium formed.

20 設有兩個分別都有兩子兩女的家庭。若所有子女都已達適婚年齡，則明顯地會有 4 個不同的組合使該 8 名男女組合成 4 對夫婦。問若有三個擁有兩子兩女，且達適婚年齡的家庭，則有多少方法組合這 12 名男女成為 6 對夫婦?
There are 2 families, each with 2 sons and 2 daughters, all having reached an age ready for getting married. Clearly there are 4 different ways to match up the eight men and women as four couples. How many different ways are there to match up the twelve men and women as six couples if there are 3 families, each with 2 sons and 2 daughters and all having reached an age ready for getting married?

21 將 1, 2, 3, 4, 5, 6 這六個數配置成一個倒三角的形狀，使得每個數均為其上一層的左右兩個數的差。其中一個例子如下 ：

5 2 6
3 4
1

透過回答下列問題，以找出所有上述的配置方法(兩個配置方法互為對方的反射對稱影像會被視為相同的配置方法)：

(i) 設第一層中唯一的一個數為 A_1，A_2 和 B_2 為第二層中的兩數使 $A_2 > B_2$。而在第三層中，最接近 A_2 的數則為 A_3 及 B_3，其中 $A_3 > B_3$。問 A_3 必定是多少?

(ii) A_1, B_2, B_3 這三個數必定是怎樣的組合?

(iii) 使用 (i) 及 (ii) 找出所有配置的方法。

The six numbers 1, 2, 3, 4, 5, 6 are to be placed as an inverted triangular array so that each number is the difference between the two numbers to its left and right on the next higher level. One example is :

5 2 6
3 4
1

You are going to find ALL such arrays (regarding two such arrays as the same if one is the mirror image of the other) by answering the following questions.

(i) Let A_1 be the single number on the first level. Let A_2 and B_2 be the two numbers on the second level with $A_2 > B_2$. Let the two numbers on the third level nearest to A_2 be A_3 and B_3 with $A_3 > B_3$. What must A_3 be?

(ii) What must A_1, B_2, B_3 collectively be?

(iii) Use (i) and (ii) to find ALL such arrays.

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