第廿六屆香港青少年數學精英選拔賽

The 26th Hong Kong Mathematical High Achievers Selection Contest

2023 – 2024（27 / 1 / 2024）

試題 Question Paper

甲部 (每題 2 分)

Part A (2 marks for each question)

把答案填在答題紙所提供的位置。

Write your answers in the spaces provided on the answer sheets.

1. x, y, z 是三個正整數，它們的最大公因數是 2023，而它們的最小公倍數是 91035。求 $x + y + z$ 的最小可能值。
   x, y, z are three positive integers, their highest common factor is 2023, while their least common multiplier is 91035. Find the least possible value of $x + y + z$.

2. 求$3^{3^{33}}$的個位數。
   Find the unit digit of $3^{3^{33}}$.

3. 求將 2024 寫成兩個正整數的平方差的方法數量。
   Find the number of ways to write 2024 as the difference of square of the two positive integers.

4. 2019 年 11 月 8 日是星期五，問 2046 年 6 月 30 日是星期幾？
   8th November, 2019 is Friday, which day of week is 30th June, 2046?

5. a, b, c 是互不相同的偶數，求 $a^2 + b^2 + 2c^2 - 2ac - 2bc$ 的最小可能值。
   a, b, c are distinct even integers, find the least possible value of $a^2 + b^2 + 2c^2 - 2ac - 2bc$.

6. M 是一個四位數，當 M 與 2024 相加時至少有一次進位。求有多少個數 M 滿足以上條件？
   M is a four-digit number, when M is added to 2024, there is at least one carry. How many numbers M satisfy the above conditions?

7. 設 a_n 是首 n 個正整數之積，求 $(a_1 + a_2 + a_3 + \ldots + a_{2023})$ 除以 140 的餘數。
   Let a_n denote the product of the first n positive integers, what is the remainder when $(a_1 + a_2 + a_3 + \ldots + a_{2023})$ is divided by 140?

8. 已知 a 與 b 的積比 a 與 b 的和多 142，a 和 b 均為正整數，求 $a + b$ 的最大值。
   It is given that the product of a and b is greater than the sum of a and b by 142 where a, b are positive integers. Find the greatest value of $a + b$.

9. 若$\sqrt{a} + \sqrt{b} = \sqrt{17 + 2\sqrt{30}}$，$a$ 和 b 均為正整數，且$a > b$。求 $a - b$ 的值。
   If $\sqrt{a} + \sqrt{b} = \sqrt{17 + 2\sqrt{30}}$ where a, b are positive integers and $a > b$, find the value of $a - b$.

10. 求 $\frac{20240127}{20240128\times20240126-20240127^2}$ 的值。

Find the value of $\frac{20240127}{20240128\times20240126-20240127^2}$.

11. 十個孩子排成一隊，將一定數量的糖果分給他們。已知在前 9 個孩子中，每個孩子獲得糖果數量為其餘 9 個孩子獲得糖果總數的 $\frac{1}{20}$。若第十個孩子收到的糖果數量為前九個孩子收到糖果總數的 k 倍，求 k 的值。

Ten children line up in a queue, and a certain number of sweets are distributed among all of them. It is found that among the first nine children, the number of sweets that each child received is equal to $\frac{1}{20}$ of the total number of sweets obtained by the remaining nine children. If the number of sweets that the tenth child received is equal to k times the total number of sweets received by the first nine children, find the value of k.

12. 兩個盒子裡總共放了 30 顆彈珠，且兩個盒子都不是空的。有些彈珠是紅色，其餘的則是綠色。每個盒子中的紅色彈珠數量均少於綠色彈珠數量。從每個盒子中取出一顆彈珠，若它們均都是紅色的機率是 $\frac{9}{50}$，求它們都是綠色的機率是多少？

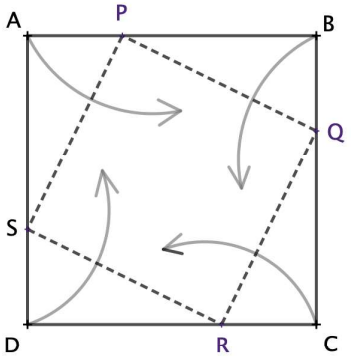
A total of 30 marbles are placed in two boxes, and neither of the boxes is empty. Some of the marbles are red, and the rest are green. In each box, the number of red marbles is less than the number of green marbles. A marble is drawn from each box. If the probability that both of them are red is $\frac{9}{50}$, what is the probability that both of them are green?

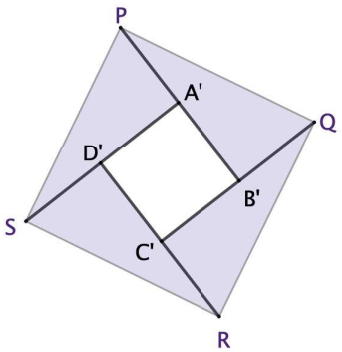
13. $ABCDEF$ 是正六邊形，其面積等於 36 cm^2。G 是 AB 上的一點，使得 $AG : GB = 1 : 3$。H 是 CD 上的一點使得 $CH : HD = 2 : 1$。求四邊形 $GBCH$ 的面積。

$ABCDEF$ is a regular hexagon and its area is equal to 36 cm^2. G is a point on AB such that $AG : GB = 1 : 3$. H is a point on CD such that $CH : HD = 2 : 1$. Find the area of the quadrilateral $GBCH$.

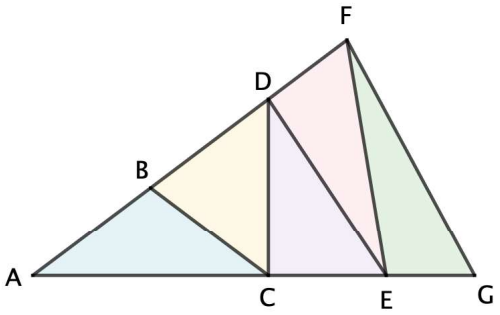
14. 一張正方形紙（一面是白色，另一面是有顏色）按下圖的方式摺。如果 $AP = BQ = CR = DS$ 並且正方形 $A'B'C'D'$ 的面積是正方形 $PQRS$ 的 $\frac{1}{5}$，求 $\frac{AP}{PB}$ 的最小值。

A piece of square paper (one side is white and the other side is coloured) is folded as shown. If $AP = BQ = CR = DS$ and the area of the square $A'B'C'D'$ is $\frac{1}{5}$ the area of square $PQRS$, find the least value of $\frac{AP}{PB}$.

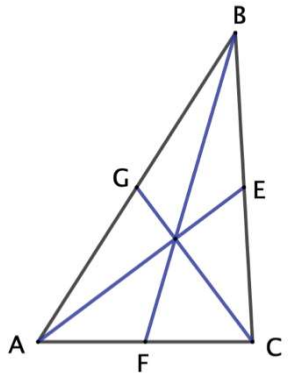




15. 在下圖中，ΔAFG 由五個面積相等的三角形組成。求 $\frac{AC}{EG}$ 。
In the figure, ΔAFG is composed of five triangles with same areas. Evaluate $\frac{AC}{EG}$.



16. 圖中，AE、BF 和 CG 為三角形 ABC 的中線。已知 $AE = 12$， $BF = 15$ 和 $CG = 9$。
求 ΔABC 的面積。
In the figure, AE, BF, CG are the medians of ΔABC.
Given that $AE = 12$, $BF = 15$ and $CG = 9$. Find the area of ΔABC.



17. 一間工廠內有兩條生產線甲和乙，生產同一種物件。甲每天生產的物件數量是乙的 4 倍。若每條生產線每天各生產多 1200 件物件，那麼甲生產的物件數量只是乙的 3 倍。原先工廠每天共生產多少件物件？
There are two production lines A and B in a factory which produce the same product. The number of items produced by A is 4 times that of B every day. If each of the production lines produces 1200 items more every day, the number of items produced by A will only be 3 times that of B. What is the total number of items produced by the factory every day originally?

18. 設 n 是正整數，且 p 和 q 是質數，滿足 $n^3 = 17p^5q^2 + n$。求 $n+3p+5q$ 的值。
Suppose n is a positive integer, p and q are prime numbers satisfying $n^3 = 17p^5q^2 + n$. Find the value of $n+3p+5q$.

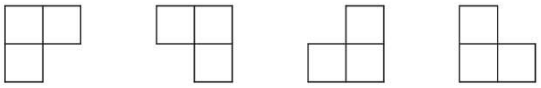
乙部 (每題 6 分)

Part B (6 marks for each question)

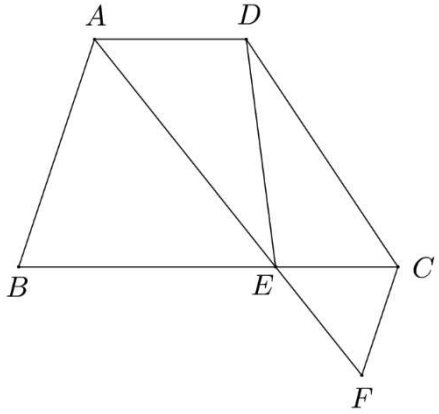
把完整的題解和答案寫在答題紙所提供的位置。

Write your full solution and answers in the spaces provided on the answer sheets.

19. 現有一個由 16 個單位正方格組成的 4×4 棋盤。我們最多可以將多少個單位正方格塗上紅色，使得沒有三個紅色單位正方格組成下圖所示的任何一個 L 形圖案？
There is a 4×4 chessboard consisting of 16 unit cells. At most how many unit cells can be coloured in red such that no three red unit cells form any of the L-shapes as shown below?



20. 設 $ABCD$ 是梯形，其中 AD 和 BC 平行。設 E 是邊 BC 上一點，使得 $BE=2EC$ 。點 F 在 AE 的延長線上，使得 CF 和 AB 平行。若 ΔCDE 的面積是 12，求 ΔCEF 的面積。
Let $ABCD$ be a trapezoid where AD and BC are parallel. Let E be a point on the side BC such that $BE=2EC$. A point F lies on the extension of AE such that CF and AB are parallel. If the area of ΔCDE is 12, find the area of ΔCEF.



21. 對於正整數 n ，記 $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$。

(a) 證明對於任何正整數 n，$n!\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)$ 是整數。

(b) 證明對於任何正整數 n， $n!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots\right)<\frac{2}{n+1}$ 。

(c) 設 $\alpha=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ ，證明 α 是無理數。

For positives integer n, denote $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

(a) Prove that for any positive integer n, $n!\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}\right)$ is an integer.

(b) Prove that for any positive integer n, $n!\left(\frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots\right)<\frac{2}{n+1}$.

(c) Let $\alpha=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ ，prove that α is an irrational number.

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